Nonlinear Systems and Control
Lecture # 22

Normal Form

Relative Degree

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x)$$

where f, g, and h are sufficiently smooth in a domain D
$f : D \rightarrow R^n$ and $g : D \rightarrow R^n$ are called vector fields on D

$$\dot{y} = \frac{\partial h}{\partial x}[f(x) + g(x)u] \stackrel{\text{def}}{=} L_f h(x) + L_g h(x)\, u$$

$$L_f h(x) = \frac{\partial h}{\partial x} f(x)$$

is the *Lie Derivative* of h with respect to f or along f

$$L_g L_f h(x) = \frac{\partial (L_f h)}{\partial x} g(x)$$

$$L_f^2 h(x) = L_f L_f h(x) = \frac{\partial (L_f h)}{\partial x} f(x)$$

$$L_f^k h(x) = L_f L_f^{k-1} h(x) = \frac{\partial (L_f^{k-1} h)}{\partial x} f(x)$$

$$L_f^0 h(x) = h(x)$$

$$\dot{y} = L_f h(x) + L_g h(x)\, u$$

$$L_g h(x) = 0 \quad \Rightarrow \quad \dot{y} = L_f h(x)$$

$$y^{(2)} = \frac{\partial (L_f h)}{\partial x} [f(x) + g(x)u] = L_f^2 h(x) + L_g L_f h(x)\, u$$

$$L_g L_f h(x) = 0 \;\Rightarrow\; y^{(2)} = L_f^2 h(x)$$

$$y^{(3)} = L_f^3 h(x) + L_g L_f^2 h(x)\, u$$

$$L_g L_f^{i-1} h(x) = 0, \;\; i = 1, 2, \ldots, \rho - 1; \quad L_g L_f^{\rho-1} h(x) \neq 0$$

$$y^{(\rho)} = L_f^{\rho} h(x) + L_g L_f^{\rho-1} h(x)\, u$$

Definition: The system

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x)$$

has relative degree ρ, $1 \leq \rho \leq n$, in $D_0 \subset D$ if $\forall\, x \in D_0$

$$L_g L_f^{i-1} h(x) = 0, \;\; i = 1, 2, \ldots, \rho - 1; \quad L_g L_f^{\rho-1} h(x) \neq 0$$

Example

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + \varepsilon(1 - x_1^2)x_2 + u, \quad y = x_1, \quad \varepsilon > 0$$

$$\dot{y} = \dot{x}_1 = x_2$$

$$\ddot{y} = \dot{x}_2 = -x_1 + \varepsilon(1 - x_1^2)x_2 + u$$

Relative degree $= 2$ over R^2

Example

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + \varepsilon(1 - x_1^2)x_2 + u, \quad y = x_2, \quad \varepsilon > 0$$

$$\dot{y} = \dot{x}_2 = -x_1 + \varepsilon(1 - x_1^2)x_2 + u$$

Relative degree $= 1$ over R^2

Example

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + \varepsilon(1 - x_1^2)x_2 + u, \quad y = x_1 + x_2^2, \quad \varepsilon > 0$$

$$\dot{y} = x_2 + 2x_2[-x_1 + \varepsilon(1 - x_1^2)x_2 + u]$$

Relative degree $= 1$ over $\{x_2 \neq 0\}$

Example: Field-controlled DC motor

$$\dot{x}_1 = -ax_1 + u, \ \dot{x}_2 = -bx_2 + k - cx_1x_3, \ \dot{x}_3 = \theta x_1 x_2, \ y = x_3$$

a, b, c, k, and θ are positive constants

$$\dot{y} = \dot{x}_3 = \theta x_1 x_2$$

$$\ddot{y} = \theta x_1 \dot{x}_2 + \theta \dot{x}_1 x_2 = (\cdot) + \theta x_2 u$$

Relative degree $= 2$ over $\{x_2 \neq 0\}$

Normal Form

Change of variables:

$$z = T(x) = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_{n-\rho}(x) \\ --- \\ h(x) \\ \vdots \\ L_f^{\rho-1}h(x) \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} \phi(x) \\ --- \\ \psi(x) \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} \eta \\ --- \\ \xi \end{bmatrix}$$

ϕ_1 to $\phi_{n-\rho}$ are chosen such that $T(x)$ is a diffeomorphism on a domain $D_0 \subset D$

$$
\begin{aligned}
\dot{\eta} &= \frac{\partial \phi}{\partial x}[f(x) + g(x)u] = f_0(\eta, \xi) + g_0(\eta, \xi)u \\
\dot{\xi}_i &= \xi_{i+1}, \quad 1 \le i \le \rho - 1 \\
\dot{\xi}_\rho &= L_f^\rho h(x) + L_g L_f^{\rho-1} h(x)\; u \\
y &= \xi_1
\end{aligned}
$$

Choose $\phi(x)$ such that $T(x)$ is a diffeomorphism and

$$
\frac{\partial \phi_i}{\partial x} g(x) = 0, \;\; \text{for } 1 \le i \le n - \rho, \; \forall\, x \in D_0
$$

Always possible (at least locally)

$$
\dot{\eta} = f_0(\eta, \xi)
$$

Theorem 13.1: Suppose the system

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x)$$

has relative degree ρ ($\leq n$) in D. If $\rho = n$, then for every $x_0 \in D$, a neighborhood N of x_0 exists such that the map $T(x) = \psi(x)$, restricted to N, is a diffeomorphism on N. If $\rho < n$, then, for every $x_0 \in D$, a neighborhood N of x_0 and smooth functions $\phi_1(x), \ldots, \phi_{n-\rho}(x)$ exist such that

$$\frac{\partial \phi_i}{\partial x} g(x) = 0, \quad \text{for } 1 \leq i \leq n - \rho$$

is satisfied for all $x \in N$ and the map $T(x) = \begin{bmatrix} \phi(x) \\ \psi(x) \end{bmatrix}$, restricted to N, is a diffeomorphism on N

Normal Form:

$$\begin{aligned}\dot{\eta} &= f_0(\eta, \xi)\\ \dot{\xi}_i &= \xi_{i+1}, \quad 1 \le i \le \rho - 1\\ \dot{\xi}_\rho &= L_f^\rho h(x) + L_g L_f^{\rho-1} h(x)\, u\\ y &= \xi_1\end{aligned}$$

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \dots & 0\\ 0 & 0 & 1 & \dots & 0\\ \vdots & & \ddots & & \vdots\\ \vdots & & & 0 & 1\\ 0 & \dots & \dots & 0 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0\\ 0\\ \vdots\\ 0\\ 1 \end{bmatrix}$$

$$C_c = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \dot{\eta} &= f_0(\eta, \xi) \\ \dot{\xi} &= A_c\xi + B_c\left[L_f^\rho h(x) + L_g L_f^{\rho-1} h(x)\; u\right] \\ y &= C_c\xi \end{aligned}$$

$$\gamma(x) = L_g L_f^{\rho-1} h(x), \quad \alpha(x) = -\frac{L_f^\rho h(x)}{L_g L_f^{\rho-1} h(x)}$$

$$\dot{\xi} = A_c\xi + B_c\gamma(x)[u - \alpha(x)]$$

If x^* is an open-loop equilibrium point at which $y = 0$; i.e., $f(x^*) = 0$ and $h(x^*) = 0$, then $\psi(x*) = 0$. Take $\phi(x^*) = 0$ so that $z = 0$ is an open-loop equilibrium point.

Zero Dynamics

$$\begin{aligned}\dot{\eta} &= f_0(\eta, \xi)\\ \dot{\xi} &= A_c\xi + B_c\gamma(x)[u - \alpha(x)]\\ y &= C_c\xi\end{aligned}$$

$$y(t) \equiv 0 \;\Rightarrow\; \xi(t) \equiv 0 \;\Rightarrow\; u(t) \equiv \alpha(x(t)) \;\Rightarrow\; \dot{\eta} = f_0(\eta, 0)$$

Definition: The equation $\dot{\eta} = f_0(\eta, 0)$ is called the *zero dynamics* of the system. The system is said to be *minimum phase* if zero dynamics have an asymptotically stable equilibrium point in the domain of interest (at the origin if $T(0) = 0$)

The zero dynamics can be characterized in the x-coordinates

$$Z^* = \{x \in D_0 \mid h(x) = L_f h(x) = \cdots = L_f^{\rho-1} h(x) = 0\}$$

$$y(t) \equiv 0 \;\Rightarrow\; x(t) \in Z^*$$

$$\Rightarrow\; u = u^*(x) \stackrel{\text{def}}{=} \alpha(x)\big|_{x \in Z^*}$$

The restricted motion of the system is described by

$$\dot{x} = f^*(x) \stackrel{\text{def}}{=} [f(x) + g(x)\alpha(x)]_{x \in Z^*}$$

Example

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + \varepsilon(1 - x_1^2)x_2 + u, \quad y = x_2$$

$$\dot{y} = \dot{x}_2 = -x_1 + \varepsilon(1 - x_1^2)x_2 + u \;\Rightarrow\; \rho = 1$$

$$y(t) \equiv 0 \;\Rightarrow x_2(t) \equiv 0 \;\Rightarrow\; \dot{x}_1 = 0$$

Non-minimum phase

Example

$$\dot{x}_1 = -x_1 + \frac{2 + x_3^2}{1 + x_3^2}\, u, \;\; \dot{x}_2 = x_3, \;\; \dot{x}_3 = x_1 x_3 + u, \;\; y = x_2$$

$$\dot{y} = \dot{x}_2 = x_3$$

$$\ddot{y} = \dot{x}_3 = x_1 x_3 + u \quad \Rightarrow \quad \rho = 2$$

$$\gamma = L_g L_f h(x) = 1, \quad \alpha = -\frac{L_f^2 h(x)}{L_g L_f h(x)} = -x_1 x_3$$

$$Z^* = \{x_2 = x_3 = 0\}$$

$$u = u^*(x) = 0 \quad \Rightarrow \quad \dot{x}_1 = -x_1$$

Minimum phase

Find $\phi(x)$ such that

$$\phi(0) = 0, \quad \frac{\partial \phi}{\partial x} g(x) = \begin{bmatrix} \frac{\partial \phi}{\partial x_1}, & \frac{\partial \phi}{\partial x_2}, & \frac{\partial \phi}{\partial x_3} \end{bmatrix} \begin{bmatrix} \frac{2+x_3^2}{1+x_3^2} \\ 0 \\ 1 \end{bmatrix} = 0$$

and

$$T(x) = \begin{bmatrix} \phi(x) & x_2 & x_3 \end{bmatrix}^T$$

is a diffeomorphism

$$\frac{\partial \phi}{\partial x_1} \cdot \frac{2 + x_3^2}{1 + x_3^2} + \frac{\partial \phi}{\partial x_3} = 0$$

$$\phi(x) = -x_1 + x_3 + \tan^{-1} x_3$$

$$T(x) = \begin{bmatrix} -x_1 + x_3 + \tan^{-1} x_3, & x_2, & x_3 \end{bmatrix}^T$$

is a global diffeomorphism

$$\eta = -x_1 + x_3 + \tan^{-1} x_3, \quad \xi_1 = x_2, \quad \xi_2 = x_3$$

$$\begin{aligned}
\dot{\eta} &= \left(-\eta + \xi_2 + \tan^{-1} \xi_2\right) \left(1 + \frac{2 + \xi_2^2}{1 + \xi_2^2} \xi_2\right) \\
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \left(-\eta + \xi_2 + \tan^{-1} \xi_2\right) \xi_2 + u \\
y &= \xi_1
\end{aligned}$$