Nonlinear Systems and Control
Lecture # 21

$\mathcal{L}_2$ Gain

&

The Small-Gain theorem

Theorem 5.4: Consider the linear time-invariant system

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du$$

where A is Hurwitz. Let $G(s) = C(sI - A)^{-1}B + D$. Then, the $\mathcal{L}_2$ gain of the system is $\sup_{\omega \in R} \|G(j\omega)\|$

Lemma: Consider the time-invariant system

$$\dot{x} = f(x, u), \qquad y = h(x, u)$$

where f is locally Lipschitz and h is continuous for all $x \in R^n$ and $u \in R^m$. Let $V(x)$ be a positive semidefinite function such that

$$\dot{V} = \frac{\partial V}{\partial x} f(x, u) \leq a(\gamma^2 \|u\|^2 - \|y\|^2), \quad a, \gamma > 0$$

Then, for each $x(0) \in R^n$, the system is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to γ. In particular

$$\|y_\tau\|_{\mathcal{L}_2} \leq \gamma \|u_\tau\|_{\mathcal{L}_2} + \sqrt{\frac{V(x(0))}{a}}$$

Proof

$$V(x(\tau)) - V(x(0)) \leq a\gamma^2 \int_0^\tau \|u(t)\|^2 \, dt - a \int_0^\tau \|y(t)\|^2 \, dt$$

$$V(x) \geq 0$$

$$\int_0^\tau \|y(t)\|^2 \, dt \leq \gamma^2 \int_0^\tau \|u(t)\|^2 \, dt + \frac{V(x(0))}{a}$$

$$\|y_\tau\|_{\mathcal{L}_2} \leq \gamma \|u_\tau\|_{\mathcal{L}_2} + \sqrt{\frac{V(x(0))}{a}}$$

Lemma 6.5: If the system

$$\dot{x} = f(x, u), \qquad y = h(x, u)$$

is output strictly passive with

$$u^T y \geq \dot{V} + \delta y^T y, \quad \delta > 0$$

then it is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to $1/\delta$

Proof

$$\begin{aligned} \dot{V} &\leq u^T y - \delta y^T y \\ &= -\frac{1}{2\delta}(u - \delta y)^T (u - \delta y) + \frac{1}{2\delta} u^T u - \frac{\delta}{2} y^T y \\ &\leq \frac{\delta}{2}\left(\frac{1}{\delta^2} u^T u - y^T y\right) \end{aligned}$$

Example

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -a x_1^3 - k x_2 + u, \quad y = x_2, \quad a, k > 0$$

$$V(x) = \tfrac{a}{4} x_1^4 + \tfrac{1}{2} x_2^2$$

$$\begin{aligned} \dot{V} &= a x_1^3 x_2 + x_2(-a x_1^3 - k x_2 + u) \\ &= -k x_2^2 + x_2 u \;=\; -k y^2 + y u \end{aligned}$$

The system is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to $1/k$

Theorem 5.5: Consider the time-invariant system

$$\dot{x} = f(x) + G(x)u, \qquad y = h(x)$$

$$f(0) = 0, \qquad h(0) = 0$$

where f and G are locally Lipschitz and h is continuous over R^n. Suppose $\exists\, \gamma > 0$ and a continuously differentiable, positive semidefinite function $V(x)$ that satisfies the Hamilton–Jacobi inequality

$$\frac{\partial V}{\partial x} f(x) + \frac{1}{2\gamma^2} \frac{\partial V}{\partial x} G(x) G^T(x) \left(\frac{\partial V}{\partial x} \right)^T + \frac{1}{2} h^T(x) h(x) \leq 0$$

$\forall\, x \in R^n$. Then, for each $x(0) \in R^n$, the system is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain $\leq \gamma$

Proof

$$\frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} G(x) u =$$

$$- \frac{1}{2}\gamma^2 \left\| u - \frac{1}{\gamma^2} G^T(x) \left(\frac{\partial V}{\partial x}\right)^T \right\|^2 + \frac{\partial V}{\partial x} f(x)$$

$$+ \frac{1}{2\gamma^2} \frac{\partial V}{\partial x} G(x) G^T(x) \left(\frac{\partial V}{\partial x}\right)^T + \frac{1}{2}\gamma^2 \|u\|^2$$

$$\dot{V} \leq \frac{1}{2}\gamma^2 \|u\|^2 - \frac{1}{2}\|y\|^2$$

Example

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

Suppose there is $P = P^T \geq 0$ that satisfies the Riccati equation

$$PA + A^T P + \frac{1}{\gamma^2} PBB^T P + C^T C = 0$$

for some $\gamma > 0$.
Verify that $V(x) = \frac{1}{2} x^T P x$ satisfies the Hamilton-Jacobi equation

The system is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to γ

Schur complement

$$\begin{bmatrix} X & Z \\ Z^T & Y \end{bmatrix} \prec 0 \Leftrightarrow Y \prec 0,\ X - ZY^{-1}Z^T \prec 0.$$

Take $X = PA + A^TP + C^TC, Z = PB, Y = -\gamma^2 I$

$$\begin{bmatrix} PA + A^TP + C^TC & PB \\ B^TP & -\gamma^2 I \end{bmatrix} \prec 0 \Leftrightarrow -\gamma^2 I \prec 0,$$

$$Q := PA + A^TP + \frac{1}{\gamma^2} PBB^TP + C^TC \prec 0$$

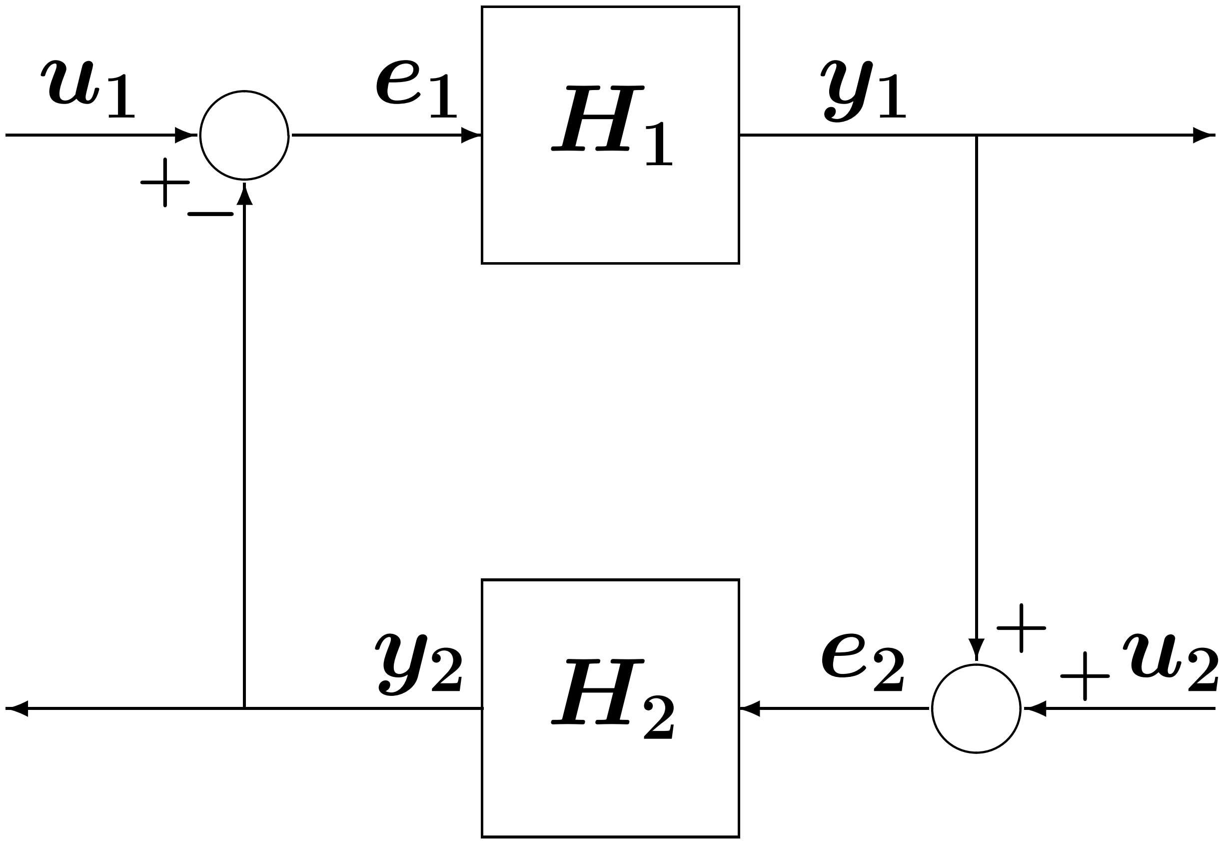
$$\Rightarrow \mathcal{H} = x^TQx < 0 \Rightarrow \dot{V} < \tfrac{1}{2}(\gamma^2\|u\|^2 - \|y\|^2)$$

The bounded real lemma The system $\Sigma = (A, B, C, D)$ is controllable and transfer function is G. Then followings are equivalent

1. $P \succeq 0, \begin{bmatrix} PA + A^T P + C^T C & PB + C^T D \\ B^T P + D^T C & D^T D - \gamma^2 I \end{bmatrix} \preceq 0$
2. for all $\omega \in R$ with $\det(j\omega I - A) \neq 0$, $G(j\omega)^* G(j\omega) \leq \gamma^2 I \Leftrightarrow \bar{\sigma}(G(j\omega)) := \|G(j\omega)\|_{2 \leftarrow 2} = \sqrt{\lambda_{\max}(G(j\omega)^* G(j\omega))} \leq \gamma$
3. $\mathcal{H}_\infty$ norm of $G(j\omega)$, i.e., $\|G\|_\infty \leq \gamma$
4. $\mathcal{L}_2$-induced norm or $\mathcal{L}_2$ gain of the system $\leq \gamma$

When $x(0) = 0$, we have $\displaystyle\sup_{u \in \mathcal{L}_2} \frac{\|y\|_{\mathcal{L}_2}}{\|u\|_{\mathcal{L}_2}} \leq \gamma$

The Small-Gain Theorem



$$\|y_{1\tau}\|_{\mathcal{L}} \le \gamma_1 \|e_{1\tau}\|_{\mathcal{L}} + \beta_1, \ \ \forall\, e_1 \in \mathcal{L}_e^m, \ \forall\, \tau \in [0, \infty)$$

$$\|y_{2\tau}\|_{\mathcal{L}} \le \gamma_2 \|e_{2\tau}\|_{\mathcal{L}} + \beta_2, \ \ \forall\, e_2 \in \mathcal{L}_e^q, \ \forall\, \tau \in [0, \infty)$$

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

Theorem: The feedback connection is finite-gain $\mathcal{L}$ stable if $\gamma_1\gamma_2 < 1$

Proof

$$e_{1\tau} = u_{1\tau} - (H_2 e_2)_\tau, \quad e_{2\tau} = u_{2\tau} + (H_1 e_1)_\tau$$

$$\begin{aligned} \|e_{1\tau}\|_\mathcal{L} &\leq \|u_{1\tau}\|_\mathcal{L} + \|(H_2 e_2)_\tau\|_\mathcal{L} \\ &\leq \|u_{1\tau}\|_\mathcal{L} + \gamma_2 \|e_{2\tau}\|_\mathcal{L} + \beta_2 \end{aligned}$$

$$
\begin{aligned}
\|e_{1\tau}\|_{\mathcal{L}} &\leq \|u_{1\tau}\|_{\mathcal{L}} + \gamma_2 \left(\|u_{2\tau}\|_{\mathcal{L}} + \gamma_1 \|e_{1\tau}\|_{\mathcal{L}} + \beta_1\right) + \beta_2 \\
&= \gamma_1\gamma_2\|e_{1\tau}\|_{\mathcal{L}} \\
&\quad + \left(\|u_{1\tau}\|_{\mathcal{L}} + \gamma_2\|u_{2\tau}\|_{\mathcal{L}} + \beta_2 + \gamma_2\beta_1\right)
\end{aligned}
$$

$$
\|e_{1\tau}\|_{\mathcal{L}} \leq \frac{1}{1-\gamma_1\gamma_2}(\|u_{1\tau}\|_{\mathcal{L}} + \gamma_2\|u_{2\tau}\|_{\mathcal{L}} + \beta_2 + \gamma_2\beta_1)
$$

$$
\|e_{2\tau}\|_{\mathcal{L}} \leq \frac{1}{1-\gamma_1\gamma_2}(\|u_{2\tau}\|_{\mathcal{L}} + \gamma_1\|u_{1\tau}\|_{\mathcal{L}} + \beta_1 + \gamma_1\beta_2)
$$

$$
\|e_\tau\|_{\mathcal{L}} \leq \|e_{1\tau}\|_{\mathcal{L}} + \|e_{2\tau}\|_{\mathcal{L}}
$$