Nonlinear Systems and Control
Lecture # 20

Input-Output Stability

Input-Output Models

$$y = Hu$$

$u(t)$ is a piecewise continuous function of t and belongs to a linear space of signals

- The space of bounded functions: $\sup_{t \geq 0} \|u(t)\| < \infty$
- The space of square-integrable functions: $\int_0^\infty u^T(t)u(t)\, dt < \infty$

Norm of a signal $\|u\|$:

- $\|u\| \geq 0$ and $\|u\| = 0 \quad \Leftrightarrow \quad u = 0$
- $\|au\| = a\|u\|$ for any $a > 0$
- Triangle Inequality: $\|u_1 + u_2\| \leq \|u_1\| + \|u_2\|$

$\mathcal{L}_p$ spaces:

$$\mathcal{L}_\infty : \quad \|u\|_{\mathcal{L}_\infty} = \sup_{t \geq 0} \|u(t)\| < \infty$$

$$\mathcal{L}_2; \quad \|u\|_{\mathcal{L}_2} = \sqrt{\int_0^\infty u^T(t)u(t)\, dt} < \infty$$

$$\mathcal{L}_p; \quad \|u\|_{\mathcal{L}_p} = \left(\int_0^\infty \|u(t)\|^p\, dt\right)^{1/p} < \infty, \quad 1 \leq p < \infty$$

Notation $\mathcal{L}_p^m$: p is the type of p-norm used to define the space and m is the dimension of u

Extended Space: $\mathcal{L}_e = \{u \mid u_\tau \in \mathcal{L}, \forall\, \tau \in [0, \infty)\}$

u_τ is a truncation of u: $u_\tau(t) = \begin{cases} u(t), & 0 \le t \le \tau \\ 0, & t > \tau \end{cases}$

$\mathcal{L}_e$ is a linear space and $\mathcal{L} \subset \mathcal{L}_e$

Example: $u(t) = t, \qquad u_\tau(t) = \begin{cases} t, & 0 \le t \le \tau \\ 0, & t > \tau \end{cases}$

$u \notin \mathcal{L}_\infty$ but $u_\tau \in \mathcal{L}_{\infty e}$

Causality: A mapping $H : \mathcal{L}_e^m \longrightarrow \mathcal{L}_e^q$ is causal if the value of the output $(Hu)(t)$ at any time t depends only on the values of the input up to time t

$$(Hu)_\tau = (Hu_\tau)_\tau$$

Definition: A mapping $H : \mathcal{L}_e^m \to \mathcal{L}_e^q$ is $\mathcal{L}$ stable if $\exists\ \alpha \in \mathcal{K}$ $\beta \geq 0$ such that

$$\|(Hu)_\tau\|_{\mathcal{L}} \leq \alpha\left(\|u_\tau\|_{\mathcal{L}}\right) + \beta, \quad \forall\ u \in \mathcal{L}_e^m \text{and } \tau \in [0, \infty)$$

It is finite-gain $\mathcal{L}$ stable if $\exists\ \gamma \geq 0$ and $\beta \geq 0$ such that

$$\|(Hu)_\tau\|_{\mathcal{L}} \leq \gamma\|u_\tau\|_{\mathcal{L}} + \beta, \quad \forall\ u \in \mathcal{L}_e^m \text{and } \tau \in [0, \infty)$$

It is small-signal $\mathcal{L}$ stable (respectively, finite-gain $\mathcal{L}$ stable) if $\exists\ r > 0$ such that the inequality is satisfied for all $u \in \mathcal{L}_e^m$ with $\sup_{0 \leq t \leq \tau} \|u(t)\| \leq r$

Example: Memoryless function $y = h(u)$

$$h(u) = a + b\tanh cu = a + b\,\frac{e^{cu} - e^{-cu}}{e^{cu} + e^{-cu}}, \quad a, b, c > 0$$

$$h'(u) = \frac{4bc}{(e^{cu} + e^{-cu})^2} \le bc \;\Rightarrow\; |h(u)| \le a + bc|u|, \;\; \forall\, u \in R$$

Finite-gain $\mathcal{L}_\infty$ stable with $\beta = a$ and $\gamma = bc$

$$h(u) = b\tanh cu, \quad |h(u)| \le bc|u|, \;\; \forall\, u \in R$$

$$\int_0^\infty |h(u(t))|^p \, dt \le (bc)^p \int_0^\infty |u(t)|^p \, dt, \quad \text{for } p \in [1, \infty)$$

Finite-gain $\mathcal{L}_p$ stable with $\beta = 0$ and $\gamma = bc$

$$h(u) = u^2$$

$$\sup_{t \geq 0} |h(u(t))| \leq \left(\sup_{t \geq 0} |u(t)| \right)^2$$

$\mathcal{L}_\infty$ stable with $\beta = 0$ and $\alpha(r) = r^2$

It is not finite-gain $\mathcal{L}_\infty$ stable. Why?

$$h(u) = \tan u$$

$$|u| \leq r < \frac{\pi}{2} \;\; \Rightarrow \;\; |h(u)| \leq \left(\frac{\tan r}{r} \right) |u|$$

Small-signal finite-gain $\mathcal{L}_p$ stable with $\beta = 0$ and $\gamma = \tan r / r$

Example: SISO causal convolution operator

$$y(t) = \int_0^t h(t-\sigma)u(\sigma)\, d\sigma, \quad h(t) = 0 \text{ for } t < 0$$

Suppose $h \in \mathcal{L}_1 \;\; \Leftrightarrow \;\; \|h\|_{\mathcal{L}_1} = \int_0^\infty |h(\sigma)|\, d\sigma < \infty$

$$\begin{aligned} |y(t)| &\leq \int_0^t |h(t-\sigma)|\, |u(\sigma)|\, d\sigma \\ &\leq \int_0^t |h(t-\sigma)|\, d\sigma \sup_{0\leq\sigma\leq\tau} |u(\sigma)| \\ &= \int_0^t |h(s)|\, ds \sup_{0\leq\sigma\leq\tau} |u(\sigma)| \end{aligned}$$

$$\|y_\tau\|_{\mathcal{L}_\infty} \leq \|h\|_{\mathcal{L}_1} \|u_\tau\|_{\mathcal{L}_\infty}, \;\; \forall\, \tau \in [0, \infty)$$

Finite-gain $\mathcal{L}_\infty$ stable

Also, finite-gain $\mathcal{L}_p$ stable for $p \in [1, \infty)$ (see textbook)

$\mathcal{L}$ Stability of State Models

$$\dot{x} = f(x, u), \quad y = h(x, u)$$

$$0 = f(0, 0), \quad 0 = h(0, 0)$$

Case 1: The origin of $\dot{x} = f(x, 0)$ is exponentially stable

$$c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2$$

$$\frac{\partial V}{\partial x} f(x, 0) \leq -c_3\|x\|^2, \quad \left\|\frac{\partial V}{\partial x}\right\| \leq c_4\|x\|$$

$$\|f(x, u) - f(x, 0)\| \leq L\|u\|, \quad \|h(x, u)\| \leq \eta_1\|x\| + \eta_2\|u\|$$

$$\forall \ \|x\| \leq r \ \text{ and } \ \|u\| \leq r_u$$

$$
\begin{aligned}
\dot{V} &= \frac{\partial V}{\partial x} f(x,0) + \frac{\partial V}{\partial x}[f(x,u) - f(x,0)] \\
&\le -c_3\|x\|^2 + c_4 L\|x\| \, \|u\| \le -\frac{c_3}{c_2} V + \frac{c_4 L}{\sqrt{c_1}} \|u\| \sqrt{V}
\end{aligned}
$$

$$
W(t) = \sqrt{V(x(t))} \;\Rightarrow\; \dot{W} \le -\left(\frac{c_3}{2c_2}\right) W + \frac{c_4 L}{2\sqrt{c_1}} \|u(t)\|
$$

$$
W(t) \le e^{-\frac{tc_3}{2c_2}} W(0) + \frac{c_4 L}{2\sqrt{c_1}} \int_0^t e^{-\frac{(t-\tau)c_3}{2c_2}} \|u(\tau)\| \, d\tau
$$

$$
\|x(t)\| \le \sqrt{\frac{c_2}{c_1}} \|x(0)\| e^{-\frac{tc_3}{2c_2}} + \frac{c_4 L}{2c_1} \int_0^t e^{-\frac{(t-\tau)c_3}{2c_2}} \|u(\tau)\| \, d\tau
$$

$$
\|y(t)\| \le k_0 \|x(0)\| e^{-at} + k_2 \int_0^t e^{-a(t-\tau)} \|u(\tau)\| \, d\tau + k_3 \, \|u(t)\|
$$

Theorem 5.1: For each $x(0)$ with $\|x(0)\| \leq r\sqrt{c_1/c_2}$, the system is small-signal finite-gain $\mathcal{L}_p$ stable for each $p \in [1, \infty]$

If the assumptions hold globally, then, for each $x(0) \in R^n$, the system is finite-gain $\mathcal{L}_p$ stable for each $p \in [1, \infty]$

Example

$$\dot{x} = -x - x^3 + u, \qquad y = \tanh x + u$$

$$V = \tfrac{1}{2}x^2 \;\Rightarrow\; x(-x - x^3) \leq -x^2$$

$$c_1 = c_2 = \tfrac{1}{2},\; c_3 = c_4 = 1, \quad L = \eta_1 = \eta_2 = 1$$

Finite-gain $\mathcal{L}_p$ stable for each $x(0) \in R$ and each $p \in [1, \infty]$

Case 2: The origin of $\dot{x} = f(x, 0)$ is asymptotically stable

Theorem 5.3: Suppose that, for all (x, u), f is locally Lipschitz and h is continuous and satisfies

$$\|h(x,u)\| \le \alpha_1(\|x\|) + \alpha_2(\|u\|) + \eta, \quad \alpha_1, \alpha_2 \in \mathcal{K}, \ \eta \ge 0$$

If $\dot{x} = f(x, u)$ is ISS, then, for each $x(0) \in R^n$, the system

$$\dot{x} = f(x,u), \quad y = h(x,u)$$

is $\mathcal{L}_\infty$ stable

Proof

$$\|x(t)\| \le \beta(\|x(0)\|, t) + \gamma\left(\sup_{0\le t\le \tau} \|u(t)\|\right), \;\; \beta \in \mathcal{KL}, \; \gamma \in \mathcal{K}$$

$$\begin{aligned} \|y(t)\| &\le \alpha_1\left(\beta(\|x(0)\|, t) + \gamma\left(\sup_{0\le t\le\tau} \|u(t)\|\right)\right) \\ &\quad + \alpha_2(\|u(t)\|) + \eta \end{aligned}$$

$$\alpha_1(a+b) \le \alpha_1(2a) + \alpha_1(2b)$$

$$\begin{aligned} \|y(t)\| &\le \alpha_1\left(2\beta(\|x(0)\|, t)\right) + \alpha_1\left(2\gamma\left(\sup_{0\le t\le\tau} \|u(t)\|\right)\right) \\ &\quad + \alpha_2(\|u(t)\|) + \eta \end{aligned}$$

$$\|y_\tau\|_{\mathcal{L}_\infty} \le \gamma_0\left(\|u_\tau\|_{\mathcal{L}_\infty}\right) + \beta_0$$

$$\gamma_0 = \alpha_1 \circ 2\gamma + \alpha_2 \quad \text{and} \quad \beta_0 = \alpha_1(2\beta(\|x(0)\|, 0)) + \eta$$

Theorem (Rephrasing of Thm 5.2): Suppose f is locally Lipschitz and h is continuous in some neighborhood of $(x = 0,\ u = 0)$. If the origin of $\dot{x} = f(x, 0)$ is asymptotically stable, then there is a constant $k_1 > 0$ such that for each $x(0)$ with $\|x(0)\| < k_1$, the system

$$\dot{x} = f(x, u), \quad y = h(x, u)$$

is small-signal $\mathcal{L}_\infty$ stable

Example

$$\dot{x}_1 = -x_1^3 + x_2, \quad \dot{x}_2 = -x_1 - x_2^3 + u, \qquad y = x_1 + x_2$$

$$V = (x_1^2 + x_2^2) \quad \Rightarrow \quad \dot{V} = -2x_1^4 - 2x_2^4 + 2x_2 u$$

$$x_1^4 + x_2^4 \geq \tfrac{1}{2}\|x\|^4$$

$$\begin{aligned}
\dot{V} &\leq -\|x\|^4 + 2\|x\||u| \\
&= -(1-\theta)\|x\|^4 - \theta\|x\|^4 + 2\|x\||u|, \quad 0 < \theta < 1 \\
&\leq -(1-\theta)\|x\|^4, \ \forall \ \|x\| \geq \left(\frac{2|u|}{\theta}\right)^{1/3}
\end{aligned}$$

ISS

$\mathcal{L}_\infty$ stable