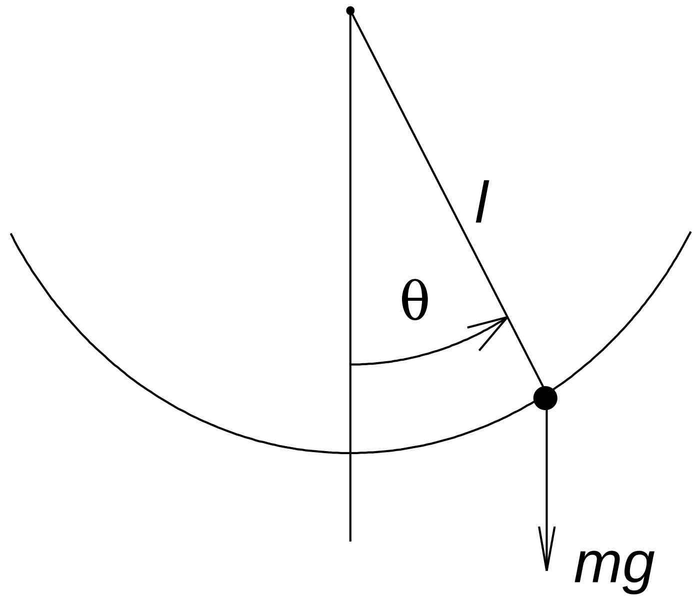
Nonlinear Systems and Control
Lecture # 2
Examples of Nonlinear Systems

Pendulum Equation



$$ml\ddot{\theta} = -mg\sin\theta - kl\dot{\theta}$$

$$x_1 = \theta, \quad x_2 = \dot{\theta}$$

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2
\end{aligned}
$$

Equilibrium Points:

$$
\begin{aligned}
0 &= x_2 \\
0 &= -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2
\end{aligned}
$$

$$(n\pi, 0) \quad \text{for} \quad n = 0, \pm 1, \pm 2, \ldots$$

Nontrivial equilibrium points at $(0, 0)$ and $(\pi, 0)$

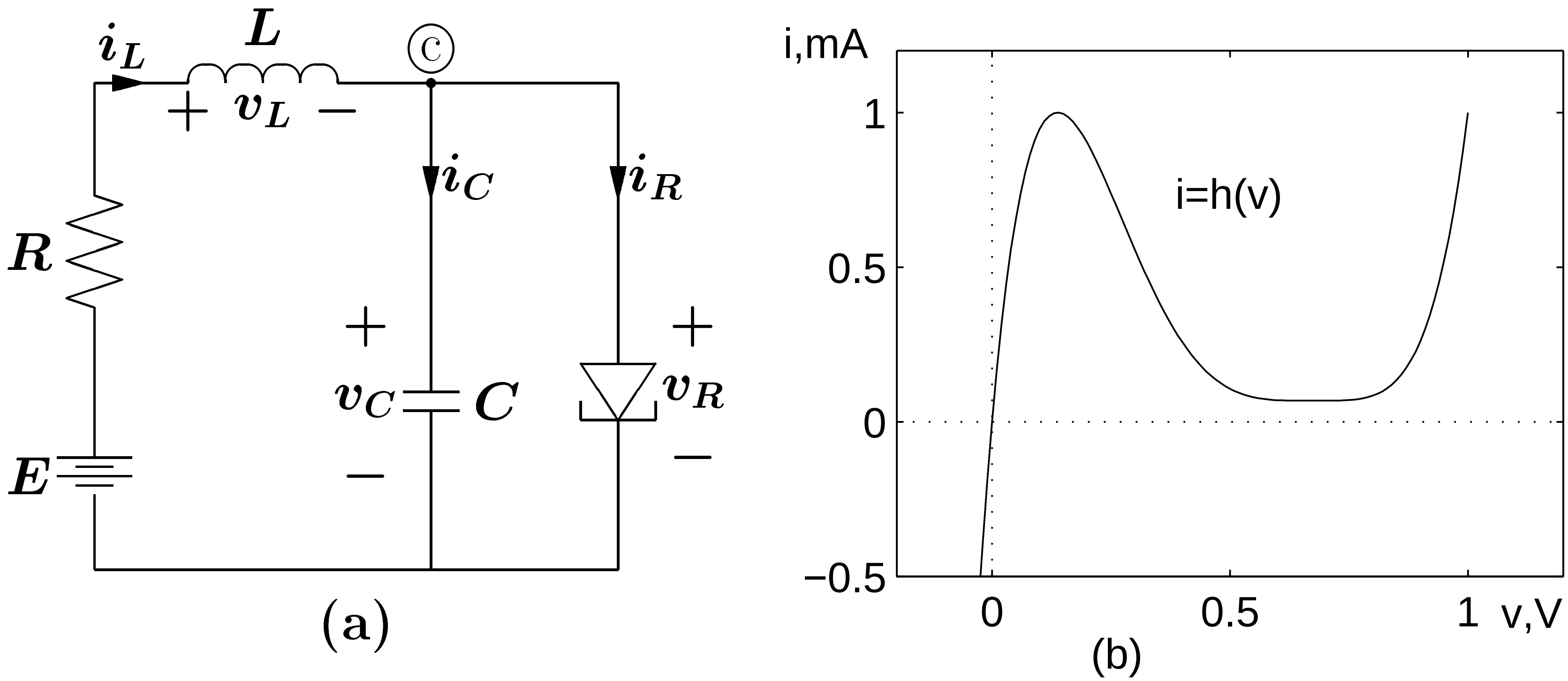
Pendulum without friction:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\frac{g}{l}\sin x_1\end{aligned}$$

Pendulum with torque input:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 + \frac{1}{ml^2}T\end{aligned}$$

Tunnel-Diode Circuit



$$i_C = C\frac{dv_C}{dt}, \qquad v_L = L\frac{di_L}{dt}$$

$$x_1 = v_C, \;\; x_2 = i_L, \quad u = E$$

$$i_C + i_R - i_L = 0 \quad \Rightarrow \quad i_C = -h(x_1) + x_2$$

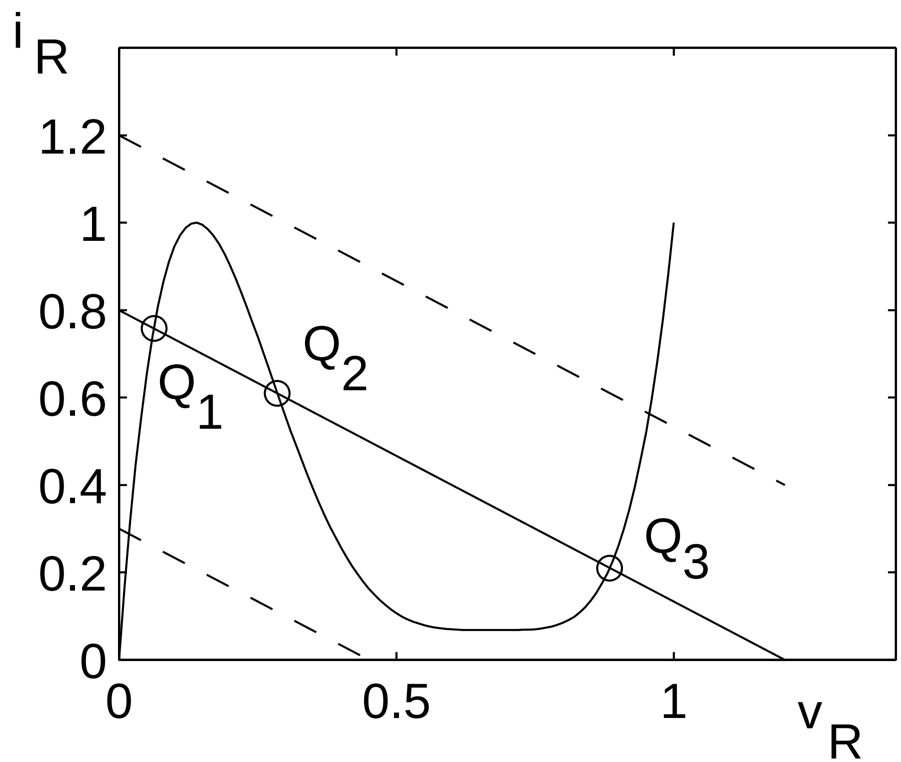
$$v_C - E + Ri_L + v_L = 0 \quad \Rightarrow \quad v_L = -x_1 - Rx_2 + u$$

$$\begin{aligned} \dot{x}_1 &= \frac{1}{C}[-h(x_1) + x_2] \\ \dot{x}_2 &= \frac{1}{L}[-x_1 - Rx_2 + u] \end{aligned}$$

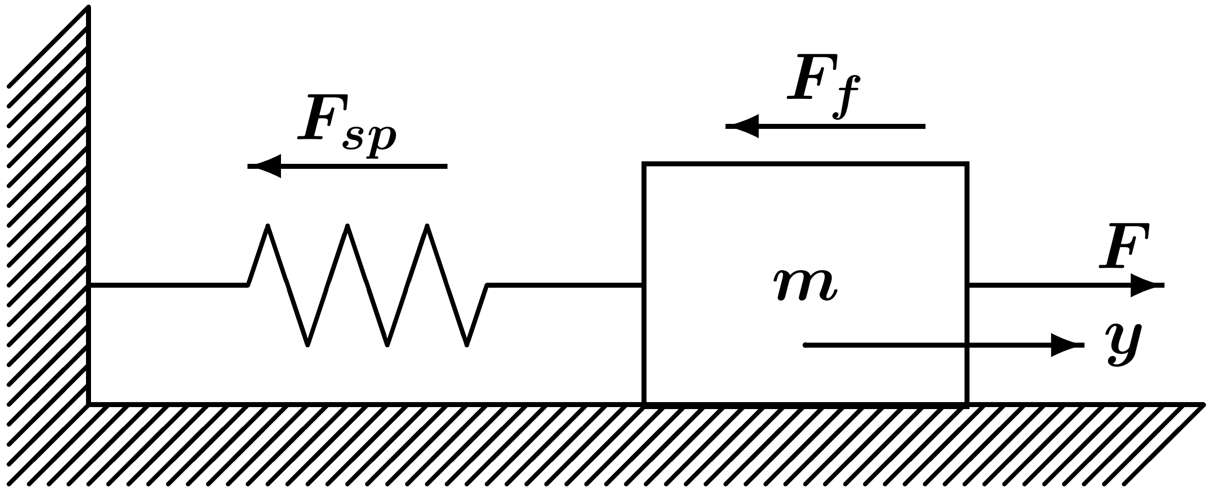
Equilibrium Points:

$$\begin{aligned} 0 &= -h(x_1) + x_2 \\ 0 &= -x_1 - Rx_2 + u \end{aligned}$$

$$h(x_1) = \frac{E}{R} - \frac{1}{R}x_1$$



Mass–Spring System



$$m\ddot{y} + F_f + F_{sp} = F$$

Sources of nonlinearity:

- Nonlinear spring restoring force $F_{sp} = g(y)$
- Static or Coulomb friction

$$F_{sp} = g(y)$$

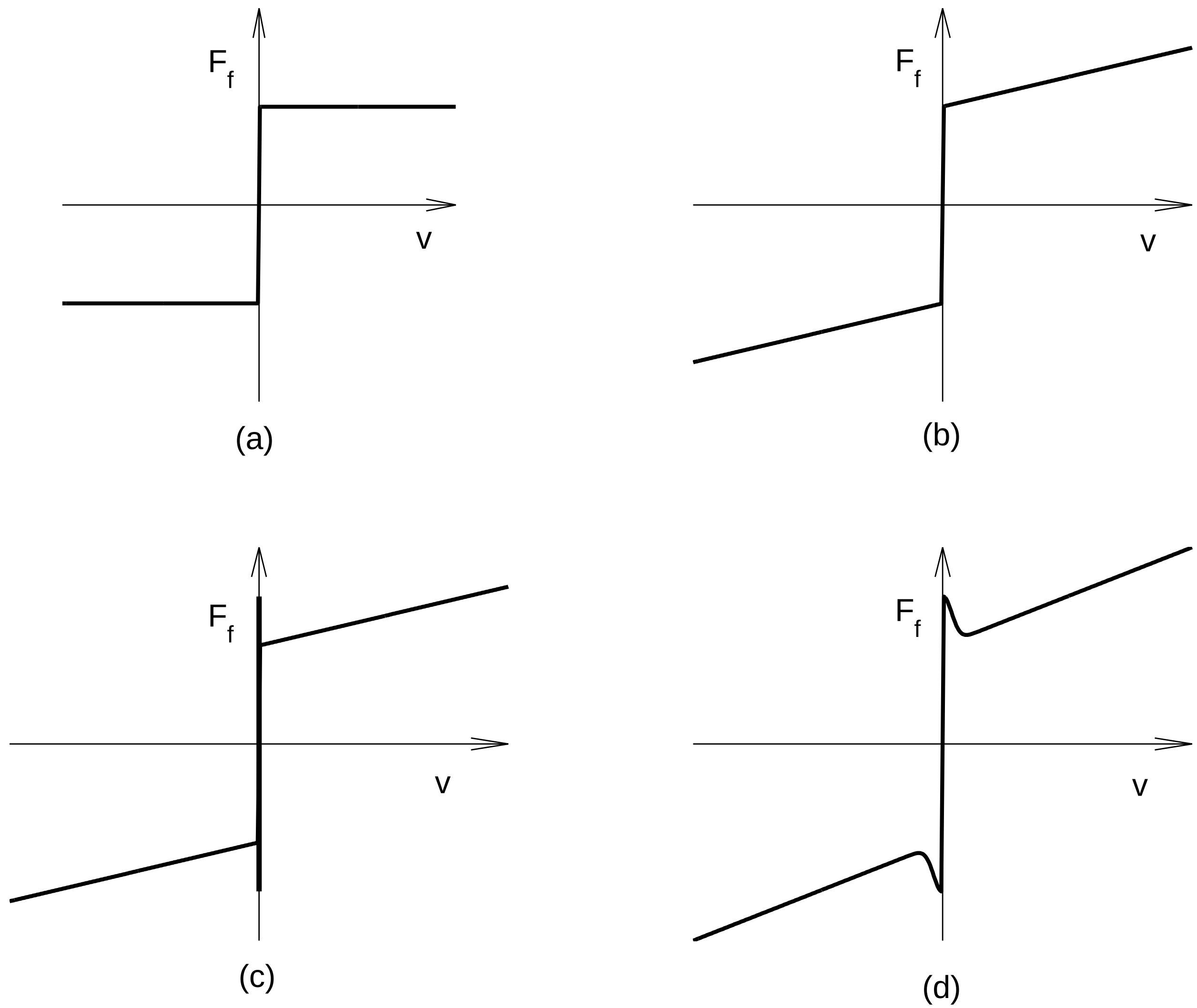
$$g(y) = k(1 - a^2y^2)y, \quad |ay| < 1 \quad \text{(softening spring)}$$

$$g(y) = k(1 + a^2y^2)y \quad \text{(hardening spring)}$$

F_f may have components due to static, Coulomb, and viscous friction

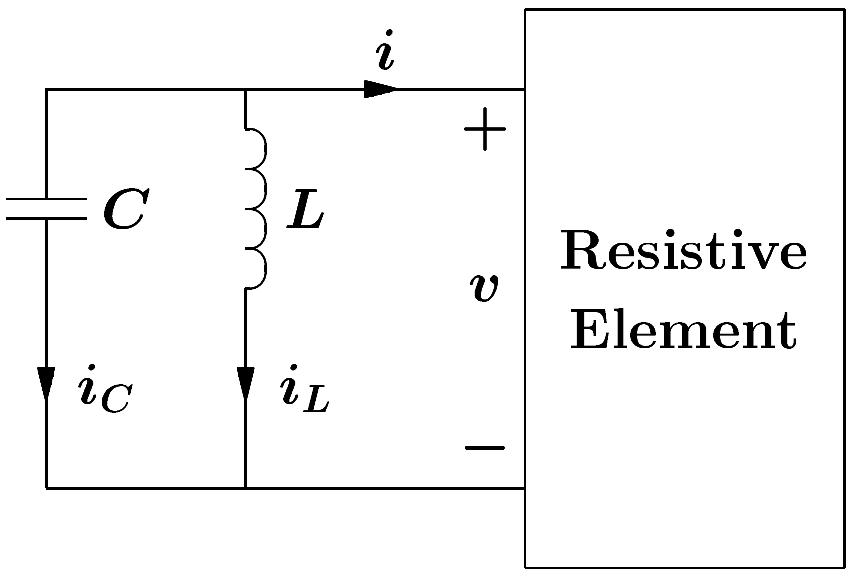
When the mass is at rest, there is a static friction force F_s that acts parallel to the surface and is limited to $\pm\mu_s mg$ $(0 < \mu_s < 1)$. F_s takes whatever value, between its limits, to keep the mass at rest

Once motion has started, the resistive force F_f is modeled as a function of the sliding velocity $v = \dot{y}$

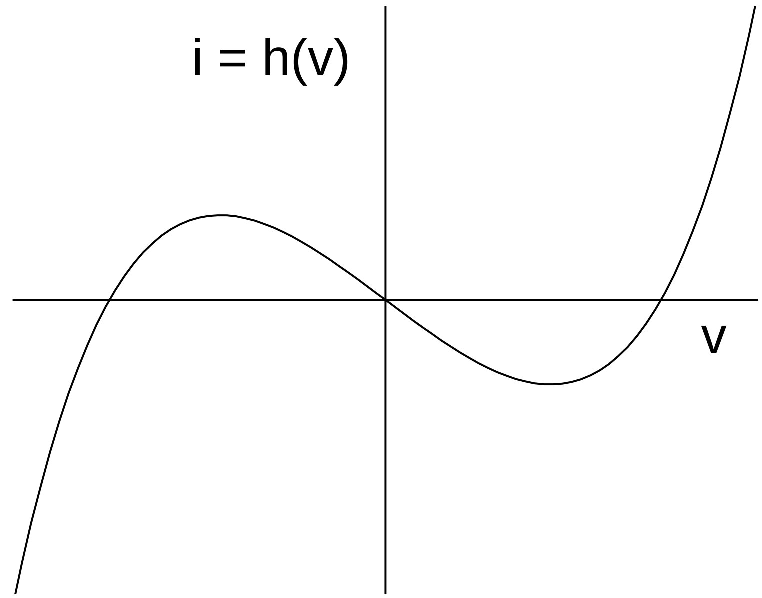


(a) Coulomb friction; (b) Coulomb plus linear viscous friction; (c) static, Coulomb, and linear viscous friction; (d) static, Coulomb, and linear viscous friction—Stribeck effect

Negative-Resistance Oscillator



(a)



(b)

$$h(0) = 0, \quad h'(0) < 0$$

$$h(v) \to \infty \text{ as } v \to \infty, \text{ and } h(v) \to -\infty \text{ as } v \to -\infty$$

$$i_C + i_L + i = 0$$

$$C\frac{dv}{dt} + \frac{1}{L}\int_{-\infty}^{t} v(s)\, ds + h(v) = 0$$

Differentiating with respect to t and multiplying by L:

$$CL\frac{d^2v}{dt^2} + v + Lh'(v)\frac{dv}{dt} = 0$$

$$\tau = t/\sqrt{CL}$$

$$\frac{dv}{d\tau} = \sqrt{CL}\frac{dv}{dt}, \qquad \frac{d^2v}{d\tau^2} = CL\frac{d^2v}{dt^2}$$

Denote the derivative of v with respect to τ by $\dot{v}$

$$\ddot{v} + \varepsilon h'(v)\dot{v} + v = 0, \quad \varepsilon = \sqrt{L/C}$$

Special case: Van der Pol equation

$$h(v) = -v + \tfrac{1}{3}v^3$$

$$\ddot{v} - \varepsilon(1 - v^2)\dot{v} + v = 0$$

State model: $x_1 = v, \quad x_2 = \dot{v}$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - \varepsilon h'(x_1)x_2 \end{aligned}$$

Another State Model: $z_1 = i_L, \quad z_2 = v_C$

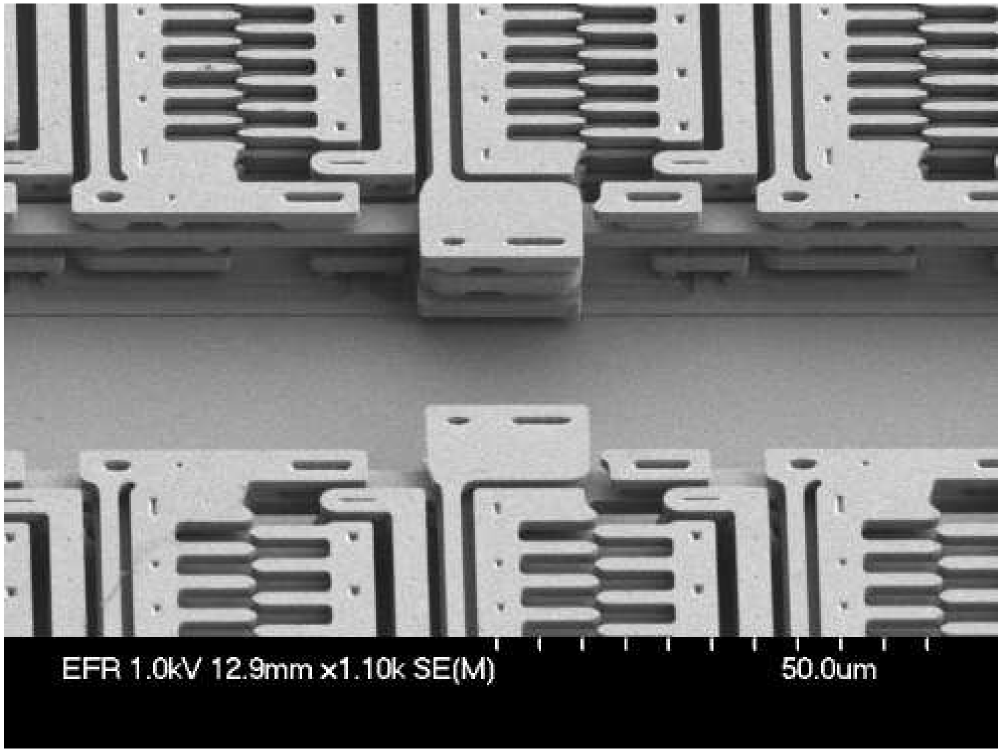
$$\begin{aligned} \dot{z}_1 &= \frac{1}{\varepsilon} z_2 \\ \dot{z}_2 &= -\varepsilon[z_1 + h(z_2)] \end{aligned}$$

Change of variables: $z = T(x)$

$$\begin{aligned} x_1 &= v = z_2 \\ x_2 &= \frac{dv}{d\tau} = \sqrt{CL}\frac{dv}{dt} = \sqrt{\frac{L}{C}}[-i_L - h(v_C)] \\ &= \varepsilon[-z_1 - h(z_2)] \end{aligned}$$

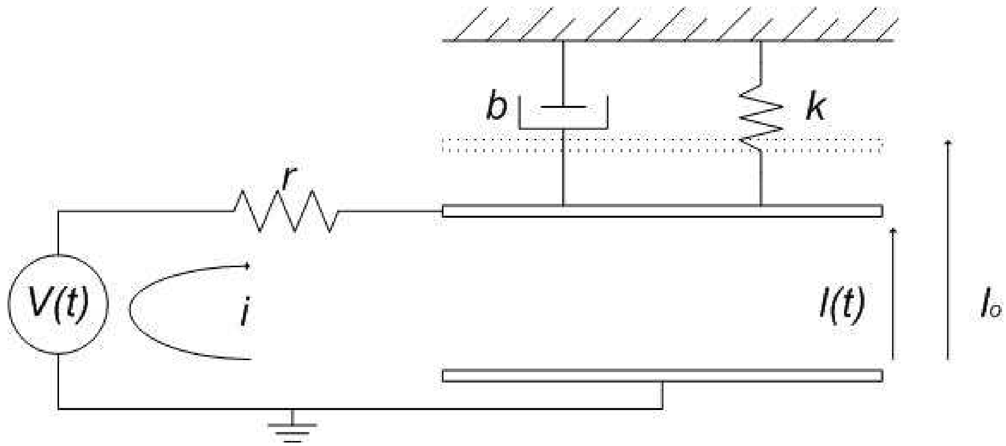
$$T(x) = \begin{bmatrix} -h(x_1) - \frac{1}{\varepsilon}x_2 \\ x_1 \end{bmatrix}, \quad T^{-1}(z) = \begin{bmatrix} z_2 \\ -\varepsilon z_1 - \varepsilon h(z_2) \end{bmatrix}$$

Electrostatically Actuated MEMS:



(from www.memx.com)

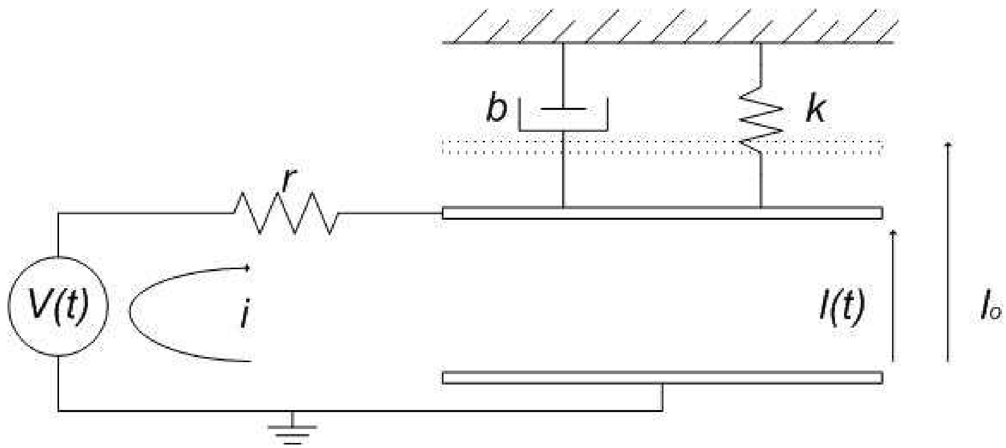
Electrostatically Actuated MEMS:



$v(t)$ input control voltage; $Q(t)$ charge of the device; $i(t)$ current; $l(t)$ air gap; $l_0(t)$ zero voltage gap; A plate area; ϵ permittivity in the gap

Attractive electrostatic force: $F(t) = \frac{Q(t)^2}{2\epsilon A}$

Electrostatically Actuated MEMS:



$$
\begin{aligned}
m\ddot{l}(t) &= -b\dot{l}(t) - k(l(t) - l_0) - \frac{Q^2(t)}{2\epsilon A} \\
\dot{Q}(t) = i(t) &= \frac{1}{r}\left[v(t) - \frac{Q(t)l(t)}{\epsilon A}\right]
\end{aligned}
$$