Nonlinear Systems and Control
Lecture # 19

Perturbed Systems
&
Input-to-State Stability

Perturbed Systems: Nonvanishing Perturbation

Nominal System:

$$\dot{x} = f(x), \qquad f(0) = 0$$

Perturbed System:

$$\dot{x} = f(x) + g(t,x), \qquad g(t,0) \neq 0$$

Case 1: The origin of $\dot{x} = f(x)$ is exponentially stable

$$c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2$$

$$\frac{\partial V}{\partial x} f(x) \leq -c_3\|x\|^2, \quad \left\|\frac{\partial V}{\partial x}\right\| \leq c_4\|x\|$$

$$\forall\, x \in B_r = \{\|x\| \leq r\}$$

Use $V(x)$ to investigate ultimate boundedness of the perturbed system

$$\dot{V}(t,x) = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(t,x)$$

Assume

$$\|g(t,x)\| \leq \delta, \quad \forall\, t \geq 0, \;\; x \in B_r$$

$$\begin{aligned}
\dot{V}(t,x) &\leq -c_3\|x\|^2 + \left\|\frac{\partial V}{\partial x}\right\| \; \|g(t,x)\| \\
&\leq -c_3\|x\|^2 + c_4\delta\|x\| \\
&= -(1-\theta)c_3\|x\|^2 - \theta c_3\|x\|^2 + c_4\delta\|x\| \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 < \theta < 1 \\
&\leq -(1-\theta)c_3\|x\|^2, \;\; \forall\; \|x\| \geq \delta c_4/(\theta c_3) \stackrel{\text{def}}{=} \mu
\end{aligned}$$

Apply Theorem 4.18

$$\|x(t_0)\| \le \alpha_2^{-1}(\alpha_1(r)) \;\Leftrightarrow\; \|x(t_0)\| \le r\sqrt{\frac{c_1}{c_2}}$$

$$\mu < \alpha_2^{-1}(\alpha_1(r)) \;\Leftrightarrow\; \frac{\delta c_4}{\theta c_3} < r\sqrt{\frac{c_1}{c_2}} \;\Leftrightarrow\; \delta < \frac{c_3}{c_4}\sqrt{\frac{c_1}{c_2}}\theta r$$

$$b = \alpha_1^{-1}(\alpha_2(\mu)) \;\Leftrightarrow\; b = \mu\sqrt{\frac{c_2}{c_1}} \;\Leftrightarrow\; b = \frac{\delta c_4}{\theta c_3}\sqrt{\frac{c_2}{c_1}}$$

For all $\|x(t_0)\| \le r\sqrt{c_1/c_2}$, the solutions of the perturbed system are ultimately bounded by b

Example

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -4x_1 - 2x_2 + \beta x_2^3 + d(t)$$

$$\beta \geq 0, \quad |d(t)| \leq \delta, \forall\, t \geq 0$$

$$V(x) = x^T P x = x^T \begin{bmatrix} \frac{3}{2} & \frac{1}{8} \\ \frac{1}{8} & \frac{5}{16} \end{bmatrix} x \quad \text{(Lecture 13)}$$

$$\begin{aligned} \dot{V}(t,x) &= -\|x\|^2 + 2\beta x_2^2 \left(\tfrac{1}{8}x_1 x_2 + \tfrac{5}{16}x_2^2\right) \\ &\quad + 2d(t)\left(\tfrac{1}{8}x_1 + \tfrac{5}{16}x_2\right) \\ &\leq -\|x\|^2 + \frac{\sqrt{29}}{8}\beta k_2^2 \|x\|^2 + \frac{\sqrt{29}\delta}{8}\|x\| \end{aligned}$$

$$k_2 = \max_{x^T P x \le c} |x_2| = 1.8194\sqrt{c}$$

Suppose $\beta \le 8(1-\zeta)/(\sqrt{29}k_2^2) \quad (0 < \zeta < 1)$

$$\begin{aligned} \dot{V}(t,x) &\le -\zeta\|x\|^2 + \tfrac{\sqrt{29}\delta}{8}\|x\| \\ &\le -(1-\theta)\zeta\|x\|^2, \ \forall\ \|x\| \ge \tfrac{\sqrt{29}\delta}{8\zeta\theta} \stackrel{\text{def}}{=} \mu \\ &\qquad\qquad (0 < \theta < 1) \end{aligned}$$

If $\mu^2\lambda_{\max}(P) < c$, then all solutions of the perturbed system, starting in Ω_c, are uniformly ultimately bounded by

$$b = \frac{\sqrt{29}\delta}{8\zeta\theta}\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}$$

Case 2: The origin of $\dot{x} = f(x)$ is asymptotically stable

$$\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|)$$

$$\frac{\partial V}{\partial x} f(x) \le -\alpha_3(\|x\|), \qquad \left\|\frac{\partial V}{\partial x}\right\| \le k$$

$$\forall\, x \in B_r = \{\|x\| \le r\}, \qquad \alpha_i \in \mathcal{K},\ i = 1, 2, 3$$

$$\begin{aligned}
\dot{V}(t,x) &\le -\alpha_3(\|x\|) + \left\|\frac{\partial V}{\partial x}\right\| \, \|g(t,x)\| \\
&\le -\alpha_3(\|x\|) + \delta k \\
&\le -(1-\theta)\alpha_3(\|x\|) - \theta\alpha_3(\|x\|) + \delta k \\
&\qquad\qquad\qquad\qquad\qquad 0 < \theta < 1 \\
&\le -(1-\theta)\alpha_3(\|x\|),\ \forall\, \|x\| \ge \alpha_3^{-1}\left(\frac{\delta k}{\theta}\right) \stackrel{\text{def}}{=} \mu
\end{aligned}$$

Apply Theorem 4.18

$$\mu < \alpha_2^{-1}(\alpha_1(r)) \quad \Leftrightarrow \quad \alpha_3^{-1}\left(\frac{\delta k}{\theta}\right) < \alpha_2^{-1}(\alpha_1(r))$$

$$\Leftrightarrow \quad \delta < \frac{\theta \alpha_3(\alpha_2^{-1}(\alpha_1(r)))}{k}$$ Compare with $\delta < \frac{c_3}{c_4}\sqrt{\frac{c_1}{c_2}}\theta r$

Example

$$\dot{x} = -\frac{x}{1+x^2}$$

$$V(x) = x^4 \quad \Rightarrow \quad \frac{\partial V}{\partial x}\left[-\frac{x}{1+x^2}\right] = -\frac{4x^4}{1+x^2}$$

$$\alpha_1(|x|) = \alpha_2(|x|) = |x|^4; \quad \alpha_3(|x|) = \frac{4|x|^4}{1+|x|^2}; \quad k = 4r^3$$

The origin is globally asymptotically stable

$$\frac{\theta\alpha_3(\alpha_2^{-1}(\alpha_1(r)))}{k} = \frac{\theta\alpha_3(r)}{k} = \frac{r\theta}{1+r^2}$$

$$\frac{r\theta}{1+r^2} \to 0 \text{ as } r \to \infty$$

$$\dot{x} = -\frac{x}{1+x^2} + \delta, \quad \delta > 0$$

$$\delta > \tfrac{1}{2} \Rightarrow \lim_{t\to\infty} x(t) = \infty$$

Input-to-State Stability (ISS)

Definition: The system $\dot{x} = f(x,u)$ is input-to-state stable if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for any initial state $x(t_0)$ and any bounded input $u(t)$

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) + \gamma\left(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|\right)$$

ISS of $\dot{x} = f(x,u)$ implies

- BIBS stability
- $x(t)$ is ultimately bounded by a class $\mathcal{K}$ function of $\sup_{t \geq t_0} \|u(t)\|$
- $\lim_{t\to\infty} u(t) = 0 \quad \Rightarrow \lim_{t\to\infty} x(t) = 0$
- The origin of $\dot{x} = f(x,0)$ is GAS

Theorem (Special case of Thm 4.19): Let $V(x)$ be a continuously differentiable function such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$

$$\frac{\partial V}{\partial x} f(x,u) \leq -W_3(x), \ \ \forall \ \|x\| \geq \rho(\|u\|) > 0$$

$\forall \ x \in R^n, \ u \in R^m$, where $\alpha_1, \ \alpha_2 \in \mathcal{K}_\infty$, $\rho \in \mathcal{K}$, and $W_3(x)$ is a continuous positive definite function. Then, the system $\dot{x} = f(x,u)$ is ISS with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$

Proof: Let $\mu = \rho(sup_{\tau \geq t_0} \|u(\tau)\|)$; then

$$\frac{\partial V}{\partial x} f(x,u) \leq -W_3(x), \ \ \forall \ \|x\| \geq \mu$$

Choose ε and c such that

$$\frac{\partial V}{\partial x} f(x,u) \le -W_3(x), \quad \forall\, x \in \Lambda = \{\varepsilon \le V(x) \le c\}$$

Suppose $x(t_0) \in \Lambda$ and $x(t)$ reaches Ω_ε at $t = t_0 + T$. For $t_0 \le t \le t_0 + T$, V satisfies the conditions for the uniform asymptotic stability. Therefore, the trajectory behaves as if the origin was uniformly asymptotically stable and satisfies

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0), \quad \text{for some } \beta \in \mathcal{KL}$$

For $t \ge t_0 + T$,

$$\|x(t)\| \le \alpha_1^{-1}(\alpha_2(\mu))$$

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0) + \alpha_1^{-1}(\alpha_2(\mu)), \;\; \forall\, t \ge t_0$$

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0) + \gamma\left(\sup_{\tau \ge t_0} \|u(\tau)\|\right), \;\; \forall\, t \ge t_0$$

Since $x(t)$ depends only on $u(\tau)$ for $t_0 \le \tau \le t$, the supremum on the right-hand side can be taken over $[t_0, t]$

Example

$$\dot{x} = -x^3 + u$$

The origin of $\dot{x} = -x^3$ is globally asymptotically stable

$$V = \tfrac{1}{2}x^2$$

$$\begin{aligned} \dot{V} &= -x^4 + xu \\ &= -(1-\theta)x^4 - \theta x^4 + xu \\ &\le -(1-\theta)x^4, \;\; \forall\ |x| \ge \left(\frac{|u|}{\theta}\right)^{1/3} \end{aligned}$$

$$0 < \theta < 1$$

The system is ISS with

$$\gamma(r) = (r/\theta)^{1/3}$$

Example

$$\dot{x} = -x - 2x^3 + (1 + x^2)u^2$$

The origin of $\dot{x} = -x - 2x^3$ is globally exponentially stable

$$V = \frac{1}{2}x^2$$

$$\begin{aligned} \dot{V} &= -x^2 - 2x^4 + x(1 + x^2)u^2 \\ &= x^4 - x^2(1 + x^2) + x(1 + x^2)u^2 \\ &\le -x^4, \quad \forall \ |x| \ge u^2 \end{aligned}$$

The system is ISS with $\gamma(r) = r^2$

Example

$$\dot{x}_1 = -x_1 + x_2^2, \qquad \dot{x}_2 = -x_2 + u$$

Investigate GAS of $\quad \dot{x}_1 = -x_1 + x_2^2, \qquad \dot{x}_2 = -x_2$

$$V(x) = \tfrac{1}{2}x_1^2 + \tfrac{1}{4}x_2^4$$

$$\dot{V} = -x_1^2 + x_1x_2^2 - x_2^4 = -(x_1 - \tfrac{1}{2}x_2^2)^2 - \left(1 - \tfrac{1}{4}\right)x_2^4$$

Now $u \neq 0$,
$$\begin{aligned} \dot{V} &= -\tfrac{1}{2}(x_1 - x_2^2)^2 - \tfrac{1}{2}(x_1^2 + x_2^4) + x_2^3 u \\ &\le -\tfrac{1}{2}(x_1^2 + x_2^4) + |x_2|^3|u| \end{aligned}$$

$$\dot{V} \le -\tfrac{1}{2}(1-\theta)(x_1^2 + x_2^4) - \tfrac{1}{2}\theta(x_1^2 + x_2^4) + |x_2|^3|u|$$

$$(0 < \theta < 1)$$

$$-\frac{1}{2}\theta(x_1^2 + x_2^4) + |x_2|^3|u| \leq 0$$

if $|x_2| \geq \dfrac{2|u|}{\theta}$ or $|x_2| \leq \dfrac{2|u|}{\theta}$ and $|x_1| \geq \left(\dfrac{2|u|}{\theta}\right)^2$

if $\|x\| \geq \dfrac{2|u|}{\theta}\sqrt{1 + \left(\dfrac{2|u|}{\theta}\right)^2}$

$$\rho(r) = \frac{2r}{\theta}\sqrt{1 + \left(\frac{2r}{\theta}\right)^2}$$

$$\dot{V} \leq -\frac{1}{2}(1-\theta)(x_1^2 + x_2^4), \quad \forall\ \|x\| \geq \rho(|u|)$$

The system is ISS

Find γ

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{4}x_2^4$$

For $|x_2| \le |x_1|$, $\frac{1}{4}(x_1^2 + x_2^2) \le \frac{1}{4}x_1^2 + \frac{1}{4}x_1^2 = \frac{1}{2}x_1^2 \le V(x)$

For $|x_2| \ge |x_1|$, $\frac{1}{16}(x_1^2 + x_2^2)^2 \le \frac{1}{16}(x_2^2 + x_2^2)^2 = \frac{1}{4}x_2^4 \le V(x)$

$$\min\left\{\frac{1}{4}\|x\|^2, \ \frac{1}{16}\|x\|^4\right\} \le V(x) \le \frac{1}{2}\|x\|^2 + \frac{1}{4}\|x\|^4$$

$$\alpha_1(r) = \frac{1}{4} \ \min\left\{r^2, \ \frac{1}{4}r^4\right\}, \qquad \alpha_2(r) = \frac{1}{2}r^2 + \frac{1}{4}r^4$$

$$\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$$

$$\alpha_1^{-1}(s) = \begin{cases} 2(s)^{\frac{1}{4}}, & \text{if } s \le 1 \\ 2\sqrt{s}, & \text{if } s \ge 1 \end{cases}$$