Nonlinear Systems and Control
Lecture # 18

Boundedness
&
Ultimate Boundedness

Definition: The solutions of $\dot{x} = f(t,x)$ are

- uniformly bounded if $\exists\, c > 0$ and for every $0 < a < c$, $\exists\, \beta = \beta(a) > 0$ such that

$$\|x(t_0)\| \le a \;\Rightarrow\; \|x(t)\| \le \beta, \;\; \forall\, t \ge t_0 \ge 0$$

- uniformly ultimately bounded with ultimate bound b if $\exists\, b$ and c and for every $0 < a < c$, $\exists\, T = T(a,b) \ge 0$ such that

$$\|x(t_0)\| \le a \;\Rightarrow\; \|x(t)\| \le b, \;\; \forall\, t \ge t_0 + T$$

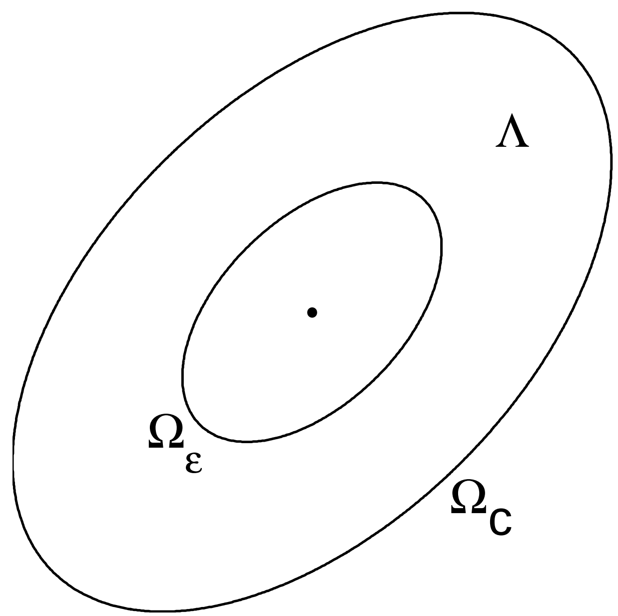
“Globally” if a can be arbitrarily large

Drop “uniformly” if $\dot{x} = f(x)$

Lyapunov Analysis: Let $V(x)$ be a cont. diff. positive definite function and suppose that the sets

$$\Omega_c = \{V(x) \leq c\},\ \Omega_\varepsilon = \{V(x) \leq \varepsilon\},\ \Lambda = \{\varepsilon \leq V(x) \leq c\}$$

are compact for some $c > \varepsilon > 0$



Suppose

$$\dot{V}(t,x) = \frac{\partial V}{\partial x} f(t,x) \leq -W_3(x), \quad \forall\, x \in \Lambda,\ \forall\, t \geq 0$$

$W_3(x)$ is continuous and positive definite

Ω_c and Ω_ε are positively invariant

$$k = \min_{x \in \Lambda} W_3(x) > 0$$

$$\dot{V}(t,x) \leq -k, \quad \forall\, x \in \Lambda,\ \forall\, t \geq t_0 \geq 0$$

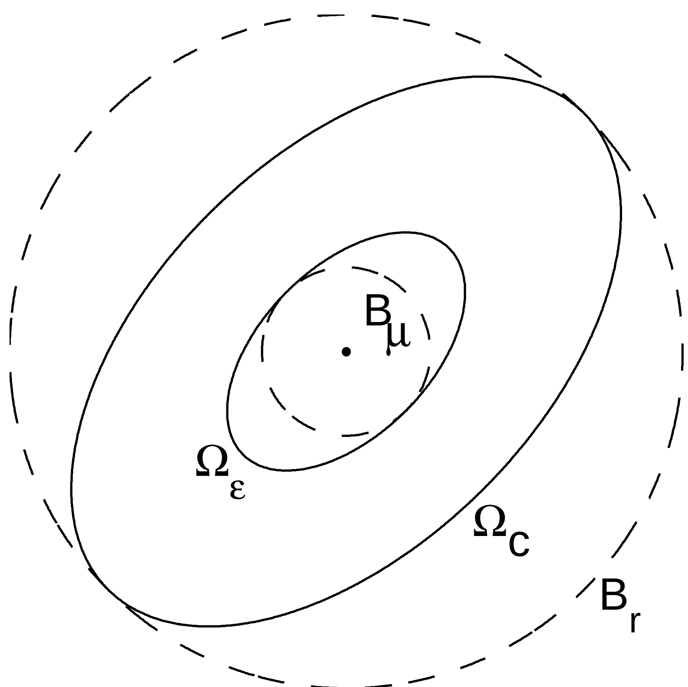
$$V(x(t)) \leq V(x(t_0)) - k(t - t_0) \leq c - k(t - t_0)$$

$x(t)$ enters the set Ω_ε within the interval $[t_0, t_0 + (c - \varepsilon)/k]$

Suppose

$$\dot{V}(t,x) \le -W_3(x), \quad \forall\, \mu \le \|x\| \le r,\ \forall\, t \ge 0$$

Choose c and ε such that $\Lambda \subset \{\mu \le \|x\| \le r\}$



Let α_1 and α_2 be class $\mathcal{K}$ functions such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$

$$V(x) \leq c \;\Rightarrow\; \alpha_1(\|x\|) \leq c \;\Leftrightarrow\; \|x\| \leq \alpha_1^{-1}(c)$$

$$c = \alpha_1(r) \;\;\Rightarrow\;\; \Omega_c \subset B_r$$

$$\|x\| \leq \mu \;\Rightarrow\; V(x) \leq \alpha_2(\mu)$$

$$\varepsilon = \alpha_2(\mu) \;\;\Rightarrow\;\; B_\mu \subset \Omega_\varepsilon$$

What is the ultimate bound?

$$V(x) \leq \varepsilon \Rightarrow \alpha_1(\|x\|) \leq \varepsilon \Leftrightarrow \|x\| \leq \alpha_1^{-1}(\varepsilon) = \alpha_1^{-1}(\alpha_2(\mu))$$

Theorem (special case of Thm 4.18): Suppose

$$\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|)$$

$$\frac{\partial V}{\partial x} f(t,x) \le -W_3(x), \ \ \forall \ \|x\| \ge \mu > 0$$

$\forall\ t \ge 0$ and $\|x\| \le r$, where $\alpha_1,\ \alpha_2 \in \mathcal{K}$, $W_3(x)$ is continuous & positive definite, and $\mu < \alpha_2^{-1}(\alpha_1(r))$. Then, for every initial state $x(t_0) \in \{\|x\| \le \alpha_2^{-1}(\alpha_1(r))\}$, there is $T \ge 0$ (dependent on $x(t_0)$ and μ) such that

$$\|x(t)\| \le \alpha_1^{-1}(\alpha_2(\mu)), \ \ \forall\ t \ge t_0 + T$$

If the assumptions hold globally and $\alpha_1 \in \mathcal{K}_\infty$, then the conclusion holds for any initial state $x(t_0)$

Remarks:

- The ultimate bound is independent of the initial state
- The ultimate bound is a class $\mathcal{K}$ function of μ; hence, the smaller the value of μ, the smaller the ultimate bound. As $\mu \longrightarrow 0$, the ultimate bound approaches zero

Example

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -(1 + x_1^2)x_1 - x_2 + M\cos\omega t, \quad M \ge 0$$

With $M = 0$, $\quad \dot{x}_2 = -(1 + x_1^2)x_1 - x_2 = -h(x_1) - x_2$

$$V(x) = x^T \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} x + 2\int_0^{x_1} (y + y^3)\, dy$$ (Example 4.5)

$$V(x) = x^T \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} x + \frac{1}{2}x_1^4 \stackrel{\text{def}}{=} x^T P x + \frac{1}{2}x_1^4$$

$$\lambda_{min}(P)\|x\|^2 \le V(x) \le \lambda_{max}(P)\|x\|^2 + \tfrac{1}{2}\|x\|^4$$

$$\alpha_1(r) = \lambda_{min}(P)r^2, \qquad \alpha_2(r) = \lambda_{max}(P)r^2 + \tfrac{1}{2}r^4$$

$$\begin{aligned}
\dot{V} &= -x_1^2 - x_1^4 - x_2^2 + (x_1 + 2x_2)M\cos\omega \\
&\le -\|x\|^2 - x_1^4 + M\sqrt{5}\|x\| \\
&= -(1-\theta)\|x\|^2 - x_1^4 - \theta\|x\|^2 + M\sqrt{5}\|x\| \\
&\qquad\qquad\qquad\qquad\qquad\qquad (0 < \theta < 1) \\
&\le -(1-\theta)\|x\|^2 - x_1^4, \;\; \forall \; \|x\| \ge M\sqrt{5}/\theta \stackrel{\text{def}}{=} \mu
\end{aligned}$$

The solutions are GUUB by

$$b = \alpha_1^{-1}(\alpha_2(\mu)) = \sqrt{\frac{\lambda_{max}(P)\mu^2 + \mu^4/2}{\lambda_{min}(P)}}$$