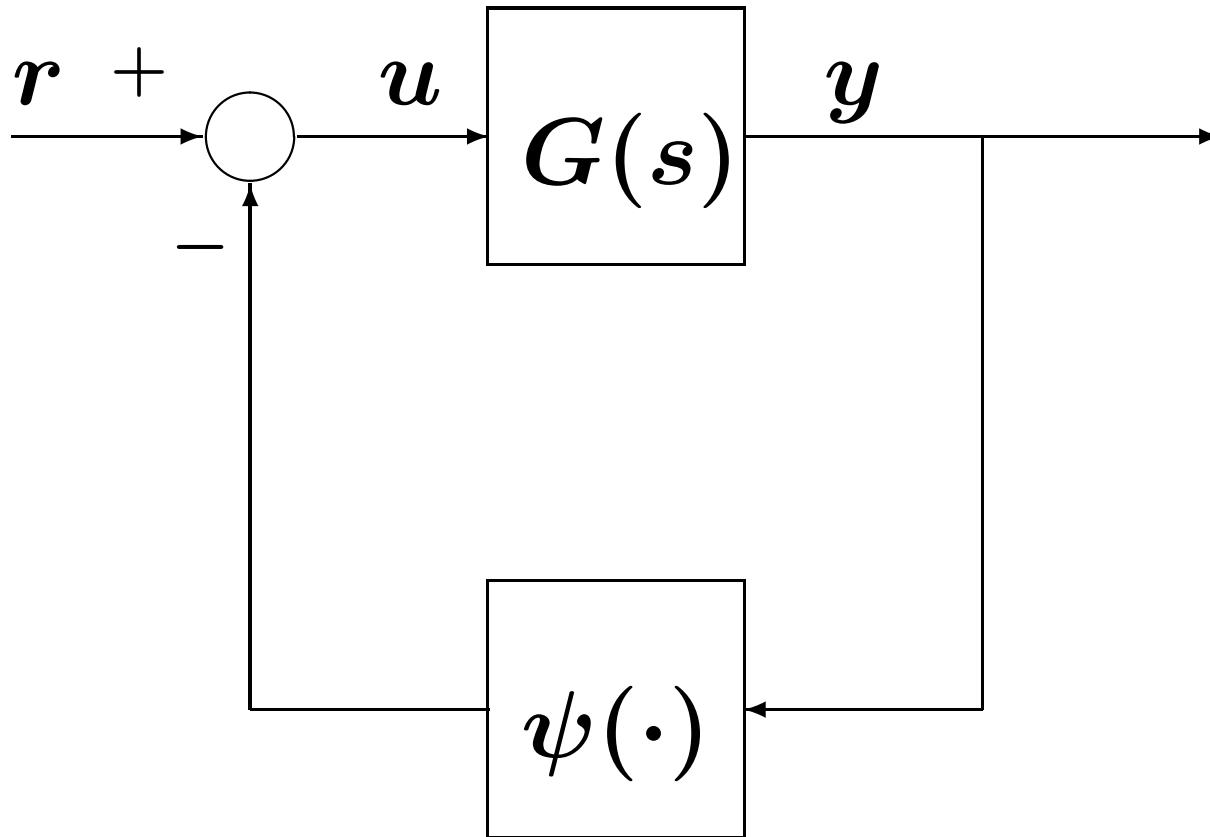


Nonlinear Systems and Control

Lecture # 17

Circle & Popov Criteria

Absolute Stability



The system is absolutely stable if (when $r = 0$) the origin is globally asymptotically stable for all memoryless time-invariant nonlinearities in a given sector

Circle Criterion

Suppose $G(s) = C(sI - A)^{-1}B + D$ is SPR and $\psi \in [0, \infty]$

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du \\ u &= -\psi(y)\end{aligned}$$

By the KYP Lemma, $\exists P = P^T > 0, L, W, \varepsilon > 0$

$$\begin{aligned}PA + A^T P &= -L^T L - \varepsilon P \\ PB &= C^T - L^T W \\ W^T W &= D + D^T\end{aligned}$$

$$V(x) = \frac{1}{2}x^T Px$$

$$\begin{aligned}
\dot{V} &= \frac{1}{2}x^T P \dot{x} + \frac{1}{2}\dot{x}^T P x \\
&= \frac{1}{2}x^T (PA + A^T P)x + x^T PBu \\
&= -\frac{1}{2}x^T L^T Lx - \frac{1}{2}\varepsilon x^T Px + x^T (C^T - L^T W)u \\
&= -\frac{1}{2}x^T \textcolor{blue}{L^T Lx} - \frac{1}{2}\varepsilon x^T Px + (Cx + Du)^T u \\
&\quad - u^T Du - x^T \textcolor{blue}{L^T W u}
\end{aligned}$$

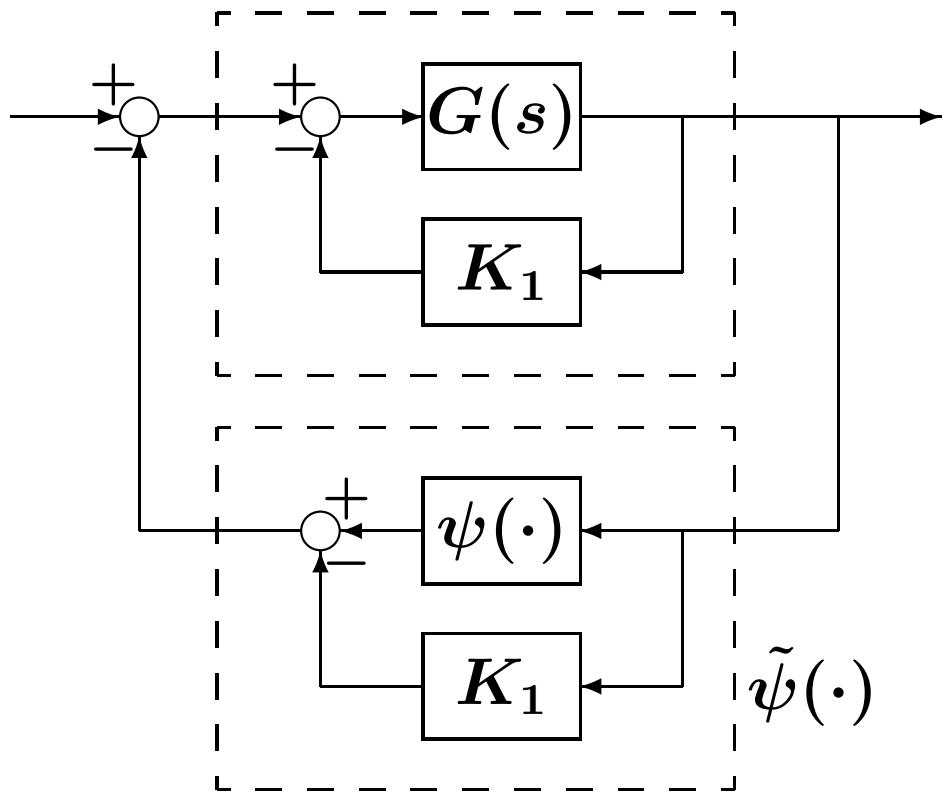
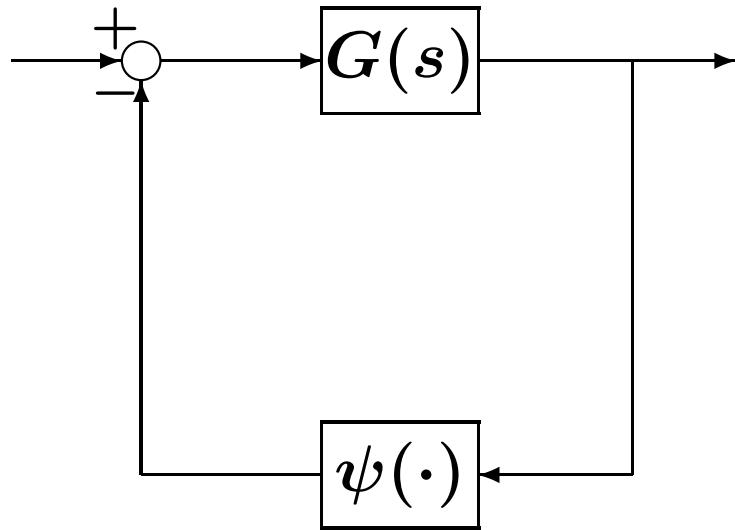
$$u^T Du = \frac{1}{2}u^T (D + D^T)u = \frac{1}{2}u^T W^T W u$$

$$\dot{V} = -\frac{1}{2}\varepsilon x^T Px - \frac{1}{2}(\textcolor{blue}{Lx + Wu})^T (\textcolor{blue}{Lx + Wu}) - y^T \psi(y)$$

$$y^T \psi(y) \geq 0 \Rightarrow \dot{V} \leq -\frac{1}{2}\varepsilon x^T Px$$

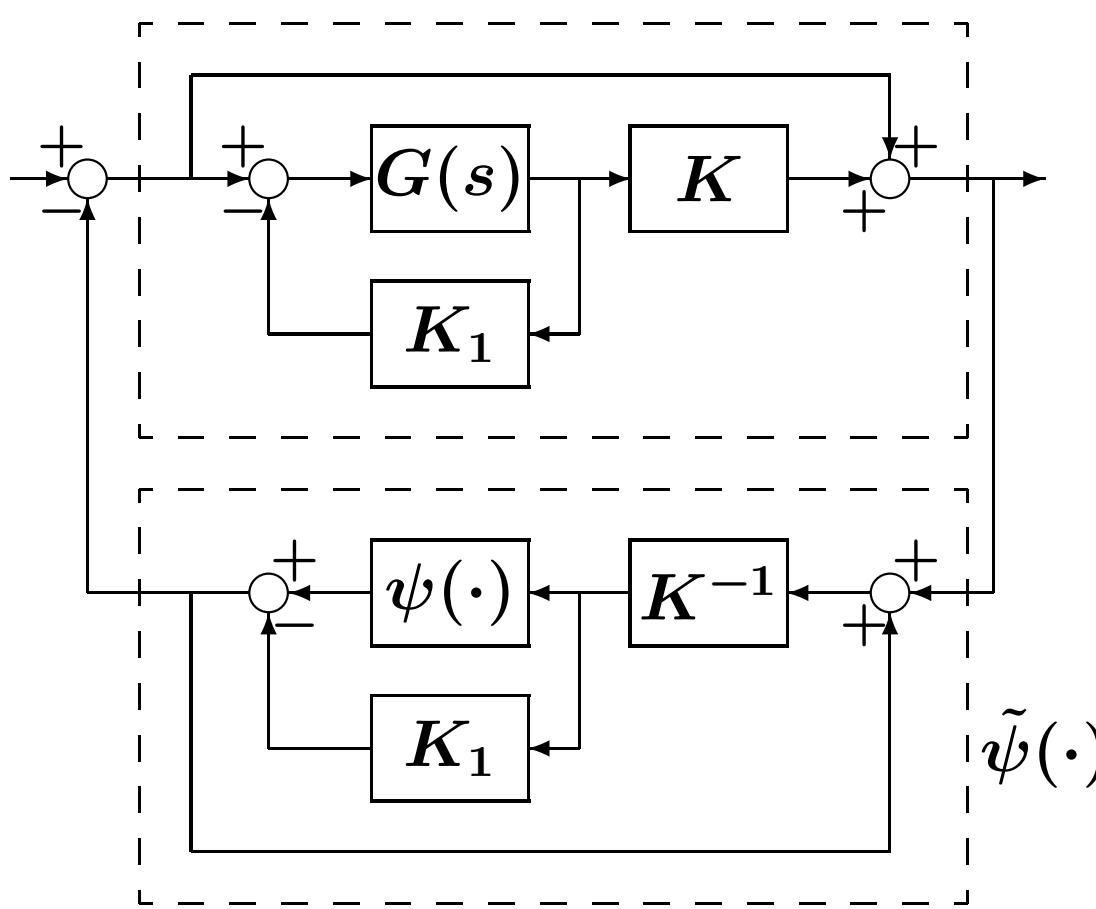
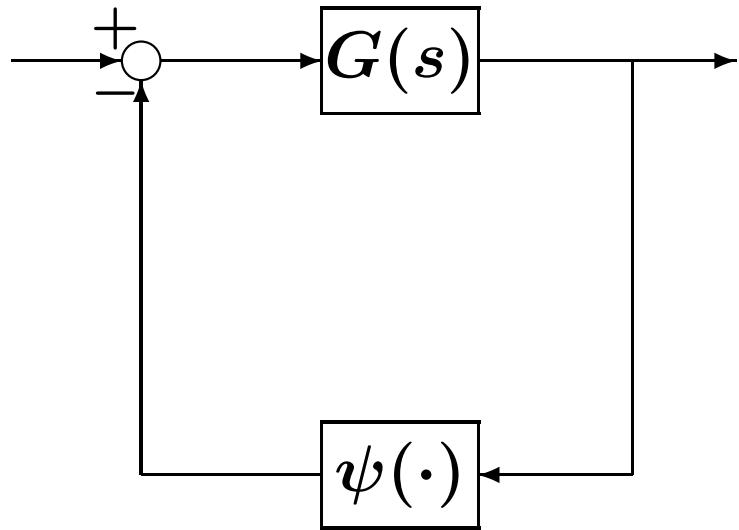
The origin is globally exponentially stable

What if $\psi \in [K_1, \infty]$?



$\tilde{\psi} \in [0, \infty]$; hence the origin is globally exponentially stable
if $G(s)[I + K_1G(s)]^{-1}$ is SPR

What if $\psi \in [K_1, K_2]$?



$\tilde{\psi} \in [0, \infty]$; hence the origin is globally exponentially stable if $I + KG(s)[I + K_1G(s)]^{-1}$ is SPR

$$I + KG(s)[I + K_1G(s)]^{-1} = [I + K_2G(s)][I + K_1G(s)]^{-1}$$

Theorem (Circle Criterion): The system is absolutely stable if

- $\psi \in [K_1, \infty]$ and $G(s)[I + K_1G(s)]^{-1}$ is SPR, or
- $\psi \in [K_1, K_2]$ and $[I + K_2G(s)][I + K_1G(s)]^{-1}$ is SPR

Scalar Case: $\psi \in [\alpha, \beta]$, $\beta > \alpha$

The system is absolutely stable if

$$\frac{1 + \beta G(s)}{1 + \alpha G(s)} \text{ is Hurwitz and}$$

$$\operatorname{Re} \left[\frac{1 + \beta G(j\omega)}{1 + \alpha G(j\omega)} \right] > 0, \quad \forall \omega \in [0, \infty]$$

Case 1: $\alpha > 0$

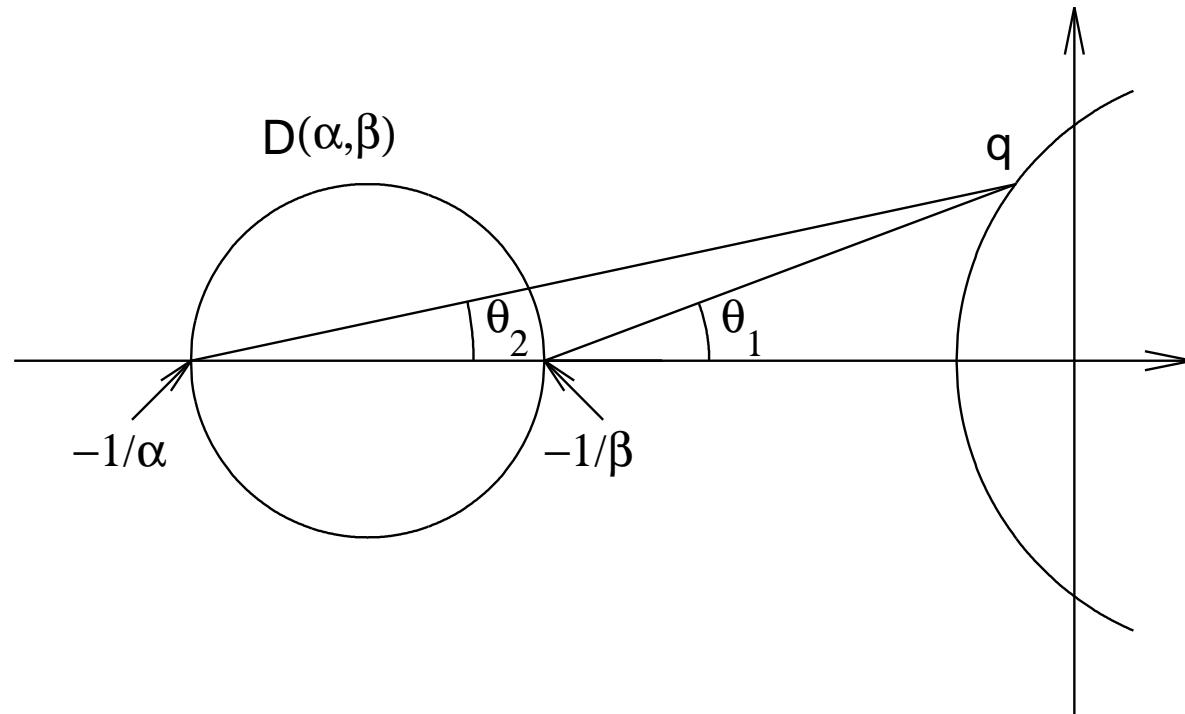
By the Nyquist criterion

$$\frac{1 + \beta G(s)}{1 + \alpha G(s)} = \frac{1}{1 + \alpha G(s)} + \frac{\beta G(s)}{1 + \alpha G(s)}$$

is Hurwitz if the Nyquist plot of $G(j\omega)$ does not intersect the point $-(1/\alpha) + j0$ and encircles it m times in the counterclockwise direction, where m is the number of poles of $G(s)$ in the open right-half complex plane

$$\frac{1 + \beta G(j\omega)}{1 + \alpha G(j\omega)} = \frac{\frac{1}{\beta} + G(j\omega)}{\frac{1}{\alpha} + G(j\omega)} \left(\frac{\beta}{\alpha} \right)$$

$$\operatorname{Re} \left[\frac{\frac{1}{\beta} + G(j\omega)}{\frac{1}{\alpha} + G(j\omega)} \right] > 0, \quad \forall \omega \in [0, \infty]$$



The system is absolutely stable if the Nyquist plot of $G(j\omega)$ does not enter the disk $D(\alpha, \beta)$ and encircles it m times in the counterclockwise direction

Case 2: $\alpha = 0$

$$1 + \beta G(s)$$

$$\operatorname{Re}[1 + \beta G(j\omega)] > 0, \quad \forall \omega \in [0, \infty]$$

$$\operatorname{Re}[G(j\omega)] > -\frac{1}{\beta}, \quad \forall \omega \in [0, \infty]$$

The system is absolutely stable if $G(s)$ is Hurwitz and the Nyquist plot of $G(j\omega)$ lies to the right of the vertical line defined by $\operatorname{Re}[s] = -1/\beta$

Case 3: $\alpha < 0 < \beta$

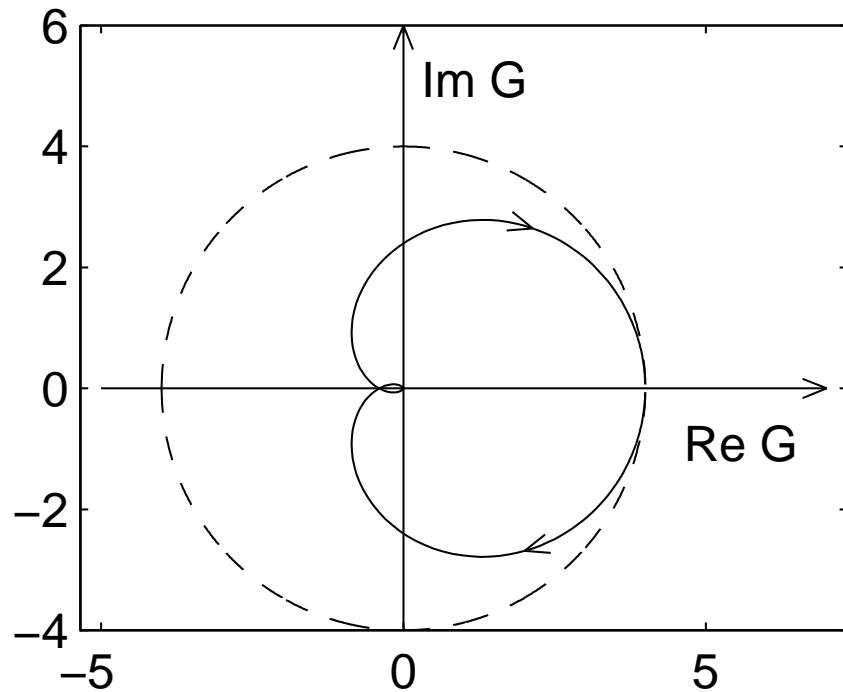
$$\operatorname{Re} \left[\frac{1 + \beta G(j\omega)}{1 + \alpha G(j\omega)} \right] > 0 \iff \operatorname{Re} \left[\frac{\frac{1}{\beta} + G(j\omega)}{\frac{1}{\alpha} + G(j\omega)} \right] < 0$$

The Nyquist plot of $G(j\omega)$ must lie inside the disk $D(\alpha, \beta)$.
The Nyquist plot cannot encircle the point $-(1/\alpha) + j0$.
From the Nyquist criterion, $G(s)$ must be Hurwitz

The system is absolutely stable if $G(s)$ is Hurwitz and the
Nyquist plot of $G(j\omega)$ lies in the interior of the disk $D(\alpha, \beta)$

Example

$$G(s) = \frac{4}{(s+1)(\frac{1}{2}s+1)(\frac{1}{3}s+1)}$$



Apply Case 3 with center $(0, 0)$ and radius = 4

Sector is $(-0.25, 0.25)$

Apply Case 3 with center $(1.5, 0)$ and radius = 2.834

Sector is $[-0.227, 0.714]$

Apply Case 2

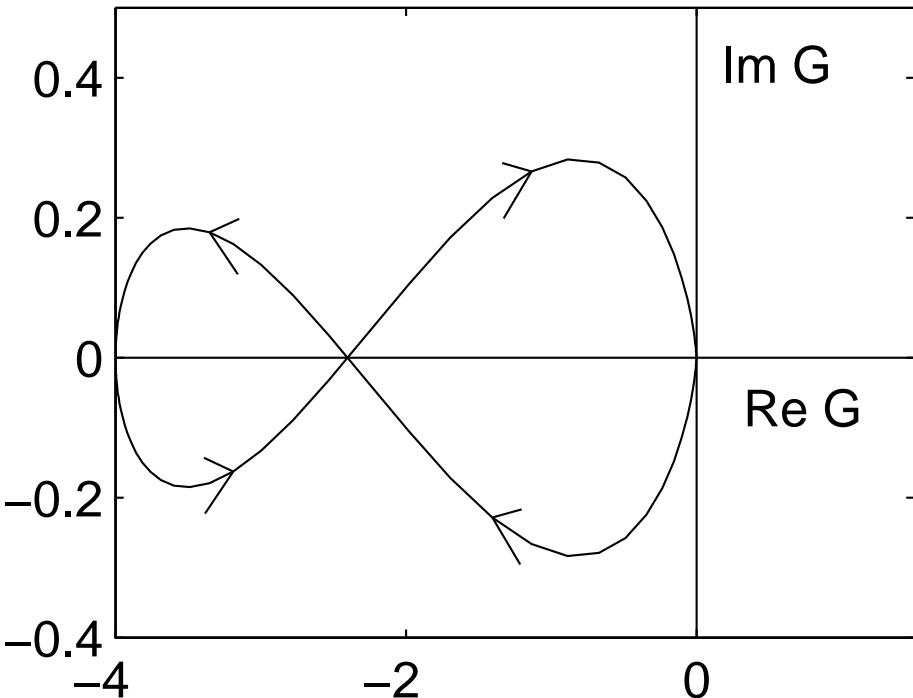
The Nyquist plot is to the right of $\text{Re}[s] = -0.857$

Sector is $[0, 1.166]$

$[0, 1.166]$ includes the saturation nonlinearity

Example

$$G(s) = \frac{4}{(s - 1)(\frac{1}{2}s + 1)(\frac{1}{3}s + 1)}$$

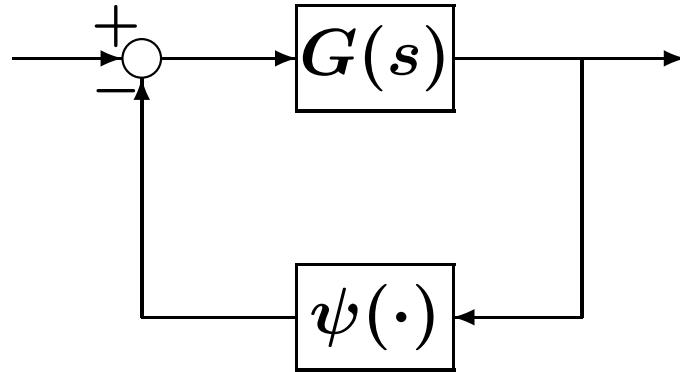


G is not Hurwitz

Apply Case 1

Center = $(-3.2, 0)$, Radius = $0.168 \Rightarrow [0.2969, 0.3298]$

Popov Criterion



$$\dot{x} = Ax + Bu$$

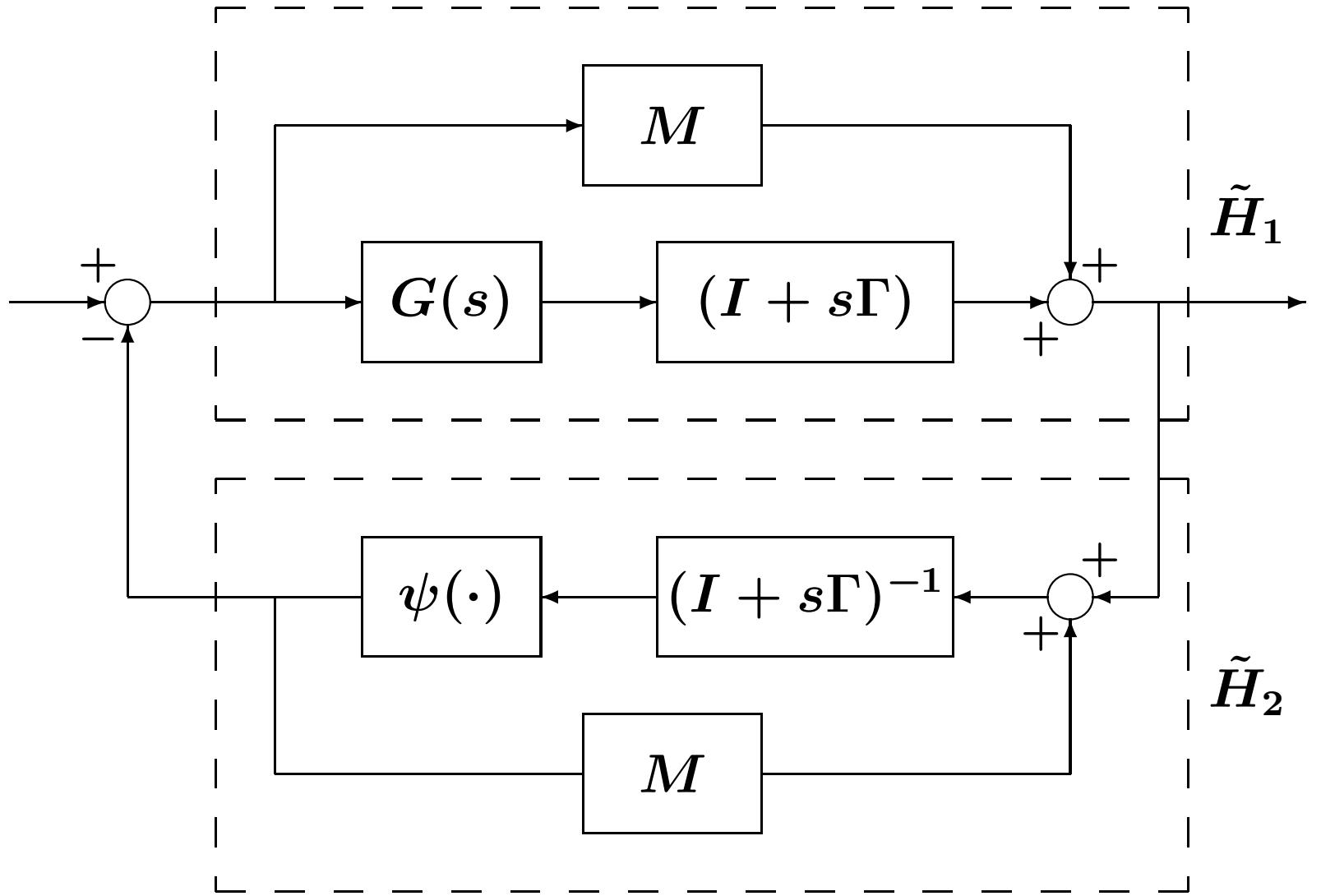
$$y = Cx$$

$$u_i = -\psi_i(y_i), \quad 1 \leq i \leq p$$

$$\psi_i \in [0, k_i], \quad 1 \leq i \leq p, \quad (0 < k_i \leq \infty)$$

$$G(s) = C(sI - A)^{-1}B$$

$$\Gamma = \text{diag}(\gamma_1, \dots, \gamma_p), \quad M = \text{diag}(1/k_1, \dots, 1/k_p)$$



Show that \tilde{H}_1 and \tilde{H}_2 are passive

$$\begin{aligned}
& M + (I + s\Gamma)G(s) \\
&= M + (I + s\Gamma)C(sI - A)^{-1}B \\
&= M + C(sI - A)^{-1}B + \Gamma C s(sI - A)^{-1}B \\
&= M + C(sI - A)^{-1}B + \Gamma C(sI - A + A)(sI - A)^{-1}B \\
&= (C + \Gamma CA)(sI - A)^{-1}B + M + \Gamma CB
\end{aligned}$$

If $M + (I + s\Gamma)G(s)$ is SPR, then \tilde{H}_1 is strictly passive with the storage function $V_1 = \frac{1}{2}x^T Px$, where P is given by the KYP equations

$$\begin{aligned}
PA + A^T P &= -L^T L - \varepsilon P \\
PB &= (C + \Gamma CA)^T - L^T W \\
W^T W &= 2M + \Gamma CB + B^T C^T \Gamma
\end{aligned}$$

\tilde{H}_2 consists of p decoupled components:

$$\gamma_i \dot{z}_i = -z_i + \frac{1}{k_i} \psi_i(z_i) + \tilde{e}_{2i}, \quad \tilde{y}_{2i} = \psi_i(z_i)$$

$$V_{2i} = \gamma_i \int_0^{z_i} \psi_i(\sigma) d\sigma$$

$$\begin{aligned}\dot{V}_{2i} &= \gamma_i \psi_i(z_i) \dot{z}_i = \psi_i(z_i) \left[-z_i + \frac{1}{k_i} \psi_i(z_i) + \tilde{e}_{2i} \right] \\ &= y_{2i} e_{2i} + \frac{1}{k_i} \psi_i(z_i) [\psi_i(z_i) - k_i z_i]\end{aligned}$$

$$\psi_i \in [0, k_i] \Rightarrow \psi_i(\psi_i - k_i z_i) \geq 0 \Rightarrow \dot{V}_{2i} \leq y_{2i} e_{2i}$$

\tilde{H}_2 is passive with the storage function

$$V_2 = \sum_{i=1}^p \gamma_i \int_0^{z_i} \psi_i(\sigma) d\sigma$$

Use $V = \frac{1}{2}x^T Px + \sum_{i=1}^p \gamma_i \int_0^{y_i} \psi_i(\sigma) d\sigma$

as a Lyapunov function candidate for the original feedback connection

$$\dot{x} = Ax + Bu, \quad y = Cx, \quad u = -\psi(y)$$

$$\begin{aligned}\dot{V} &= \frac{1}{2}x^T Px + \frac{1}{2}\dot{x}^T Px + \psi^T(y)\Gamma\dot{y} \\ &= \frac{1}{2}x^T(PA + A^TP)x + x^TPBu \\ &\quad + \psi^T(y)\Gamma C(Ax + Bu) \\ &= -\frac{1}{2}x^T L^T Lx - \frac{1}{2}\varepsilon x^T Px \\ &\quad + x^T(C^T + A^T C^T \Gamma - L^T W)u \\ &\quad + \psi^T(y)\Gamma CAx + \psi^T(y)\Gamma CBu\end{aligned}$$

$$\begin{aligned}
\dot{V} &= -\frac{1}{2}\varepsilon x^T Px - \frac{1}{2}(Lx + Wu)^T(Lx + Wu) \\
&\quad - \psi(y)^T[y - M\psi(y)] \\
&\leq -\frac{1}{2}\varepsilon x^T Px - \psi(y)^T[y - M\psi(y)]
\end{aligned}$$

$$\psi_i \in [0, k_i] \Rightarrow \psi(y)^T[y - M\psi(y)] \geq 0 \Rightarrow \dot{V} \leq -\frac{1}{2}\varepsilon x^T Px$$

The origin is globally asymptotically stable

Popov Criterion: The system is absolutely stable if, for $1 \leq i \leq p$, $\psi_i \in [0, k_i]$ and there exists a constant $\gamma_i \geq 0$, with $(1 + \lambda_k \gamma_i) \neq 0$ for every eigenvalue λ_k of A , such that $M + (I + s\Gamma)G(s)$ is strictly positive real

Scalar case

$$\frac{1}{k} + (1 + s\gamma)G(s)$$

is SPR if $G(s)$ is Hurwitz and

$$\frac{1}{k} + \operatorname{Re}[G(j\omega)] - \gamma\omega\operatorname{Im}[G(j\omega)] > 0, \quad \forall \omega \in [0, \infty)$$

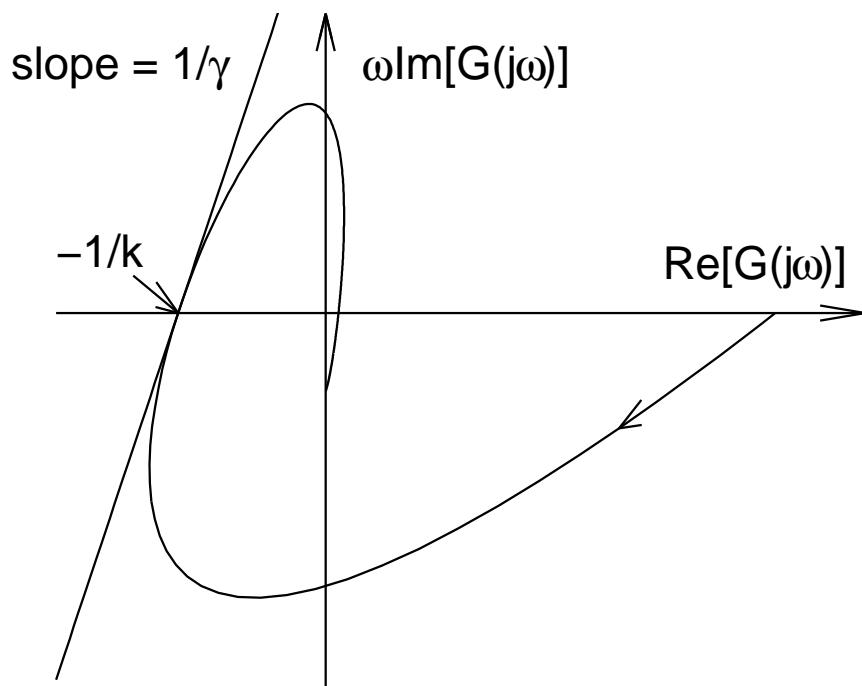
If

$$\lim_{\omega \rightarrow \infty} \left\{ \frac{1}{k} + \operatorname{Re}[G(j\omega)] - \gamma\omega\operatorname{Im}[G(j\omega)] \right\} = 0$$

we also need

$$\lim_{\omega \rightarrow \infty} \omega^2 \left\{ \frac{1}{k} + \operatorname{Re}[G(j\omega)] - \gamma\omega\operatorname{Im}[G(j\omega)] \right\} > 0$$

$$\frac{1}{k} + \operatorname{Re}[G(j\omega)] - \gamma\omega\operatorname{Im}[G(j\omega)] > 0, \quad \forall \omega \in [0, \infty)$$



Popov Plot

Example

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_2 - h(y), \quad y = x_1$$

$$\dot{x}_2 = -\alpha x_1 - x_2 - h(y) + \alpha x_1, \quad \alpha > 0$$

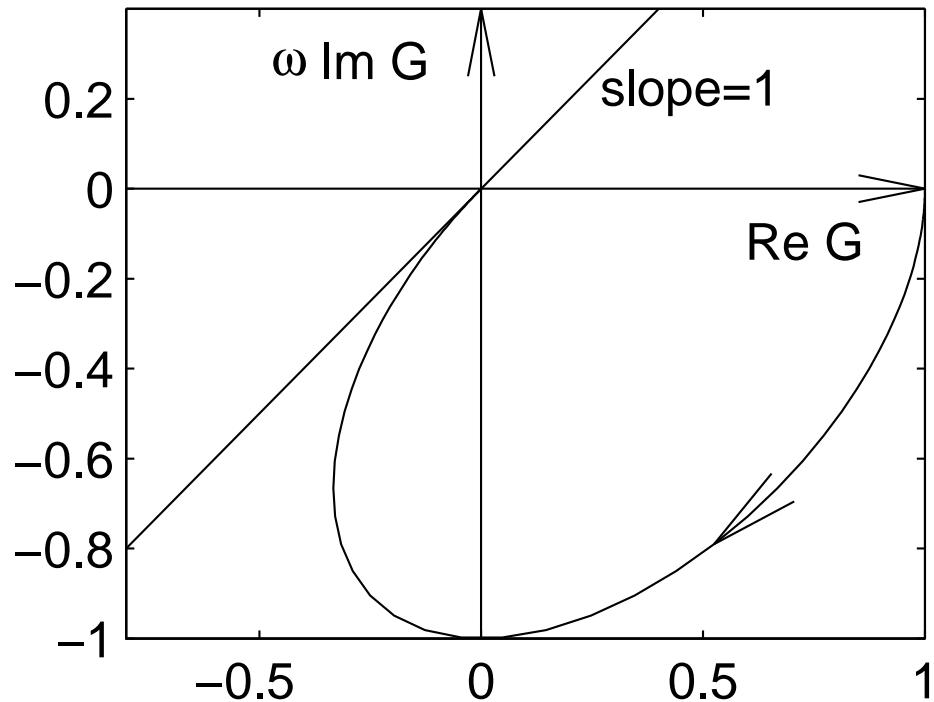
$$G(s) = \frac{1}{s^2 + s + \alpha}, \quad \psi(y) = h(y) - \alpha y$$

$$h \in [\alpha, \beta] \Rightarrow \psi \in [0, k] \quad (k = \beta - \alpha > 0)$$

$$\gamma > 1 \Rightarrow \frac{\alpha - \omega^2 + \gamma \omega^2}{(\alpha - \omega^2)^2 + \omega^2} > 0, \quad \forall \omega \in [0, \infty)$$

$$\text{and} \quad \lim_{\omega \rightarrow \infty} \frac{\omega^2(\alpha - \omega^2 + \gamma \omega^2)}{(\alpha - \omega^2)^2 + \omega^2} = \gamma - 1 > 0$$

The system is absolutely stable for $\psi \in [0, \infty]$ ($h \in [\alpha, \infty]$)



Compare with the circle criterion ($\gamma = 0$)

$$\frac{1}{k} + \frac{\alpha - \omega^2}{(\alpha - \omega^2)^2 + \omega^2} > 0, \quad \forall \omega \in [0, \infty], \quad \text{for } k < 1 + 2\sqrt{\alpha}$$