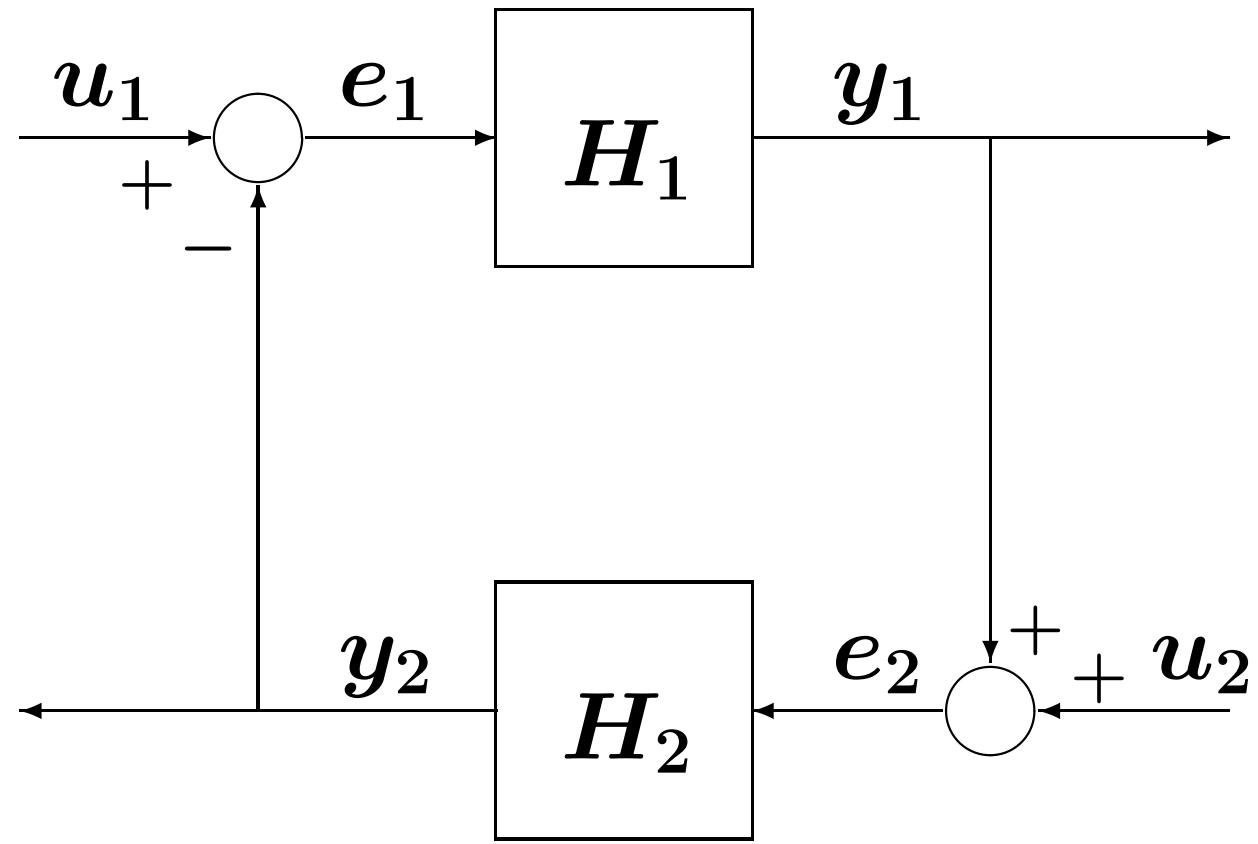


# **Nonlinear Systems and Control**

## **Lecture # 16**

### **Feedback Systems: Passivity Theorems**



$$\dot{x}_i = f_i(x_i, e_i), \quad y_i = h_i(x_i, e_i)$$

$$y_i = h_i(t, e_i)$$

## Passivity Theorems

**Theorem 6.1:** The feedback connection of two passive systems is passive

**Prove using  $V = V_1 + V_2$  as a Lyapunov func. candidate**

**Proof:** Let  $V_1(x_1)$  and  $V_2(x_2)$  be the storage functions for  $H_1$  and  $H_2$ , respectively. If  $H_i$  is memoryless, then take  $V_i = 0$ .

Since  $H_i$  is passive, we have  $e_i^T y_i \geq \dot{V}_i$

From the feedback connection,

$$e_1^T y_1 + e_2^T y_2 = (u_1 - y_2)^T y_1 + (u_2 + y_1)^T y_2 = u_1^T y_1 + u_2^T y_2$$

$$u^T y = u_1^T y_1 + u_2^T y_2 \geq \dot{V}_1 + \dot{V}_2 = \dot{V}$$

**Theorem 6.3:** Consider the feedback connection of two dynamical systems. When  $u = 0$ , the origin of the closed-loop system is **asymptotically stable** if each feedback component is either

- strictly passive, or
- output strictly passive and zero-state observable

Furthermore, if the storage function for each component is radially unbounded, the origin is globally asymptotically stable

**Proof:**  $H_1$  is SP;  $H_2$  is OSP & ZSO

$$e_1^T y_1 \geq \dot{V}_1 + \psi_1(x_1), \quad \psi_1(x_1) > 0, \quad \forall x_1 \neq 0$$

$$e_2^T y_2 \geq \dot{V}_2 + y_2^T \rho_2(y_2), \quad y_2^T \rho(y_2) > 0, \quad \forall y_2 \neq 0$$

$$e_1^T y_1 + e_2^T y_2 = (u_1 - y_2)^T y_1 + (u_2 + y_1)^T y_2 = u_1^T y_1 + u_2^T y_2$$

$$V(x) = V_1(x_1) + V_2(x_2)$$

$$\dot{V} \leq u^T y - \psi_1(x_1) - y_2^T \rho_2(y_2)$$

$$u = 0 \Rightarrow \dot{V} \leq -\psi_1(x_1) - y_2^T \rho_2(y_2)$$

$$\dot{V} = 0 \Rightarrow \textcolor{blue}{x_1 = 0} \text{ and } y_2 = 0$$

$$y_2(t) \equiv 0 \Rightarrow e_1(t) \equiv 0 \ (\& x_1(t) \equiv 0) \Rightarrow y_1(t) \equiv 0$$

$$y_1(t) \equiv 0 \Rightarrow e_2(t) \equiv 0$$

By zero-state observability of  $H_2$ :  $y_2(t) \equiv 0 \Rightarrow \textcolor{blue}{x_2(t) \equiv 0}$

Apply the invariance principle

**Theorem 6.4:** Consider the feedback connection of a strictly passive dynamical system with a passive memoryless function. When  $u = 0$ , the origin of the closed-loop system is uniformly asymptotically stable. if the storage function for the dynamical system is radially unbounded, the origin will be globally uniformly asymptotically stable

**Proof:**  $V_1$  can be shown as a positive definite function.

$$\begin{aligned}\dot{V}_1 &= \frac{\partial V_1}{\partial x_1} f_1(x_1, e_1) \leq e_1^T y_1 - \psi_1(x_1) \\ &= \underbrace{(-y_2)^T(e_2)}_{\leq 0} - \psi(x_1) \leq -\psi_1(x_1)\end{aligned}$$

## Example

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -ax_1^3 - kx_2 + e_1 \\ y_1 &= x_2 + e_1\end{aligned}\underbrace{\qquad\qquad\qquad}_{H_1}$$

$$\begin{aligned}\dot{x}_3 &= x_4 \\ \dot{x}_4 &= -bx_3 - x_4^3 + e_2 \\ y_2 &= x_4\end{aligned}\underbrace{\qquad\qquad\qquad}_{H_2}$$

$$a, b, k > 0$$

$$V_1 = \frac{1}{4}ax_1^4 + \frac{1}{2}x_2^2$$

$$\dot{V}_1 = ax_1^3x_2 - ax_1^3x_2 - kx_2^2 + x_2e_1 = -ky_1^2 + y_1e_1$$

With  $e_1 = 0$ ,  $y_1(t) \equiv 0 \Leftrightarrow x_2(t) \equiv 0 \Rightarrow x_1(t) \equiv 0$

$H_1$  is output strictly passive and zero-state observable

$$V_2 = \frac{1}{2}bx_3^2 + \frac{1}{2}x_4^2$$

$$\dot{V}_2 = bx_3x_4 - bx_3x_4 - x_4^4 + x_4e_2 = -y_2^4 + y_2e_2$$

With  $e_2 = 0$ ,  $y_2(t) \equiv 0 \Leftrightarrow x_4(t) \equiv 0 \Rightarrow x_3(t) \equiv 0$

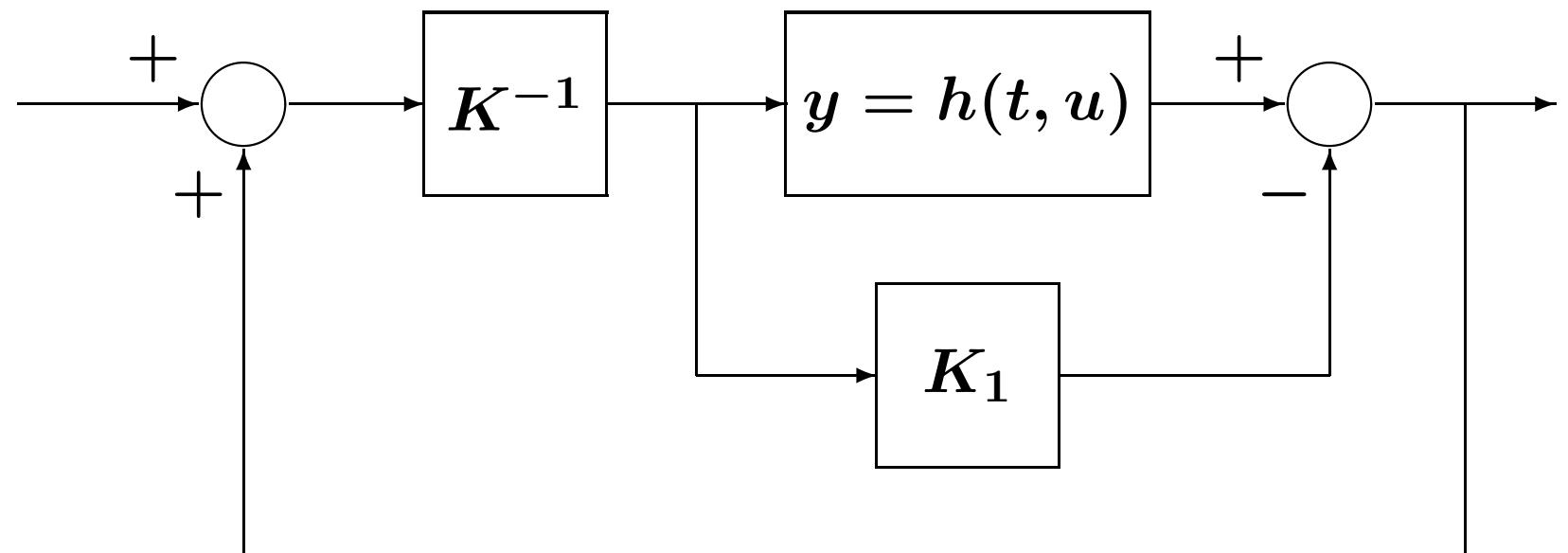
$H_2$  is output strictly passive and zero-state observable

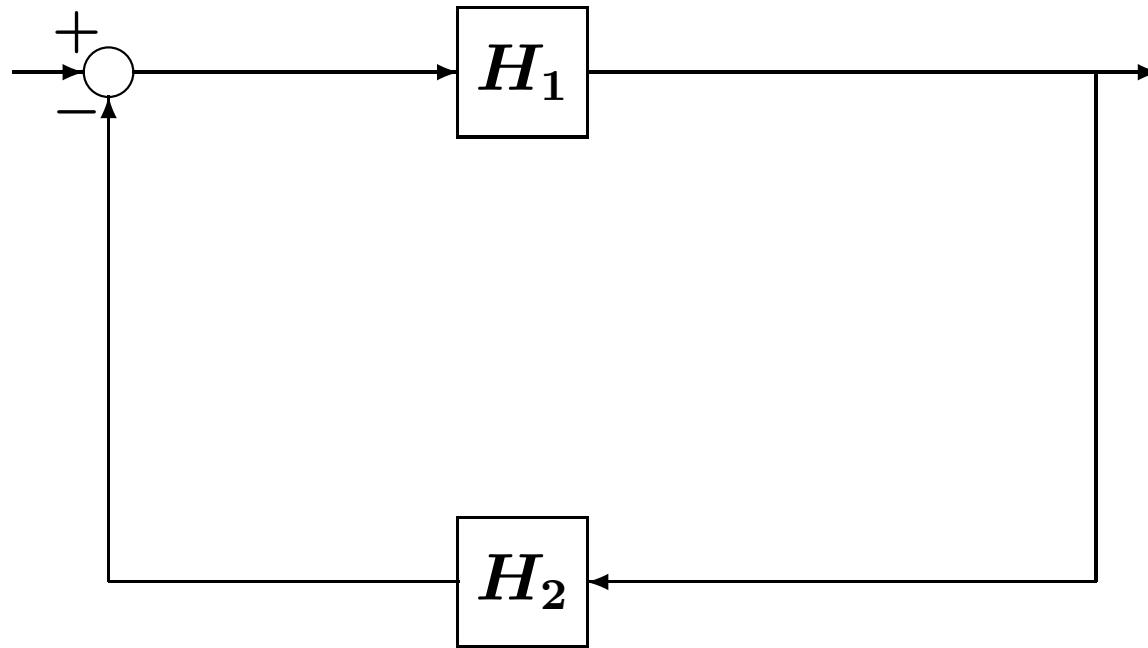
$V_1$  and  $V_2$  are radially unbounded

By Theorem 6.3, the origin is **globally asymptotically stable**

## Loop Transformations

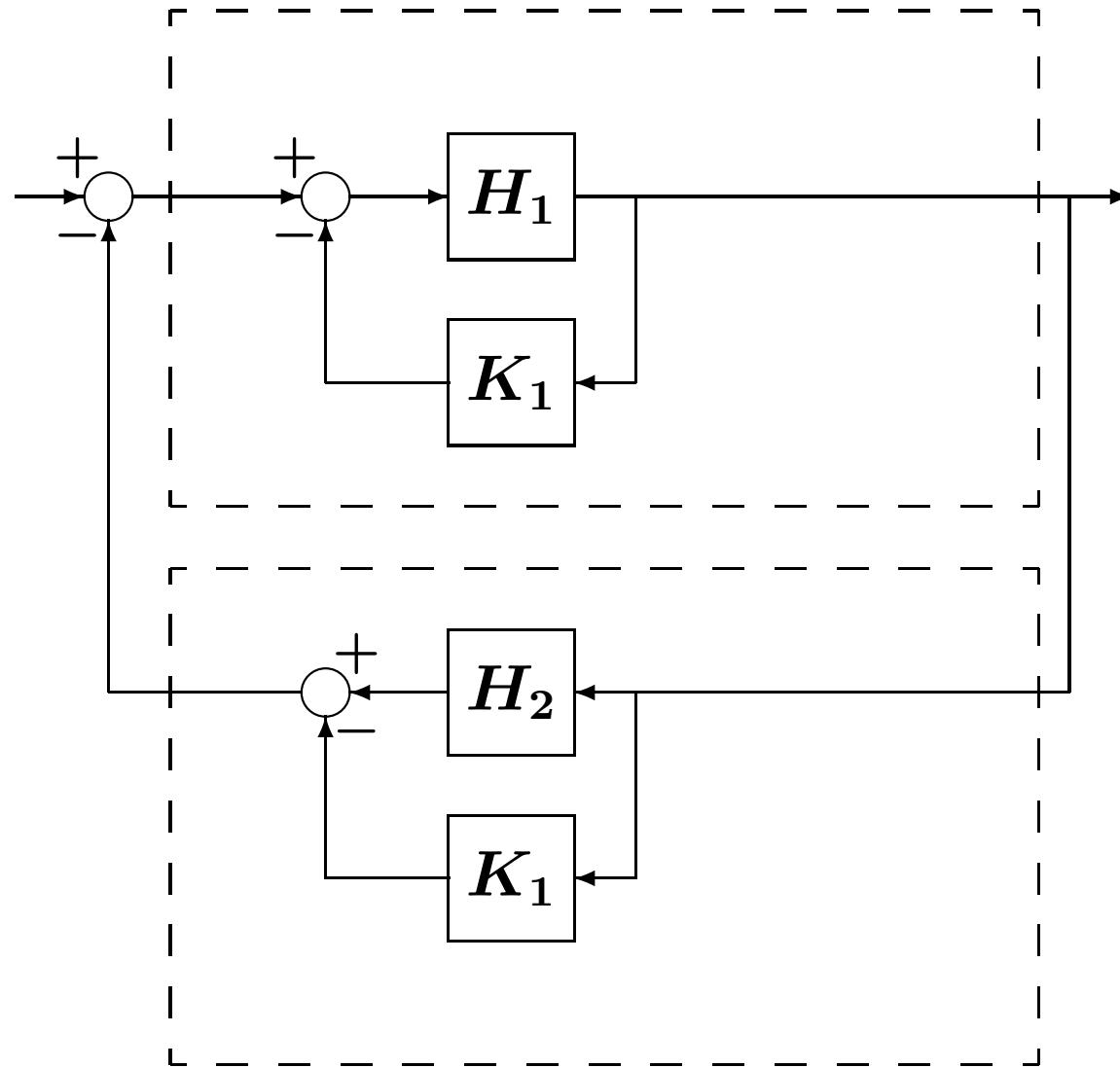
Recall that a memoryless function in the sector  $[K_1, K_2]$  can be transformed into a function in the sector  $[0, \infty]$  by input feedforward followed by output feedback

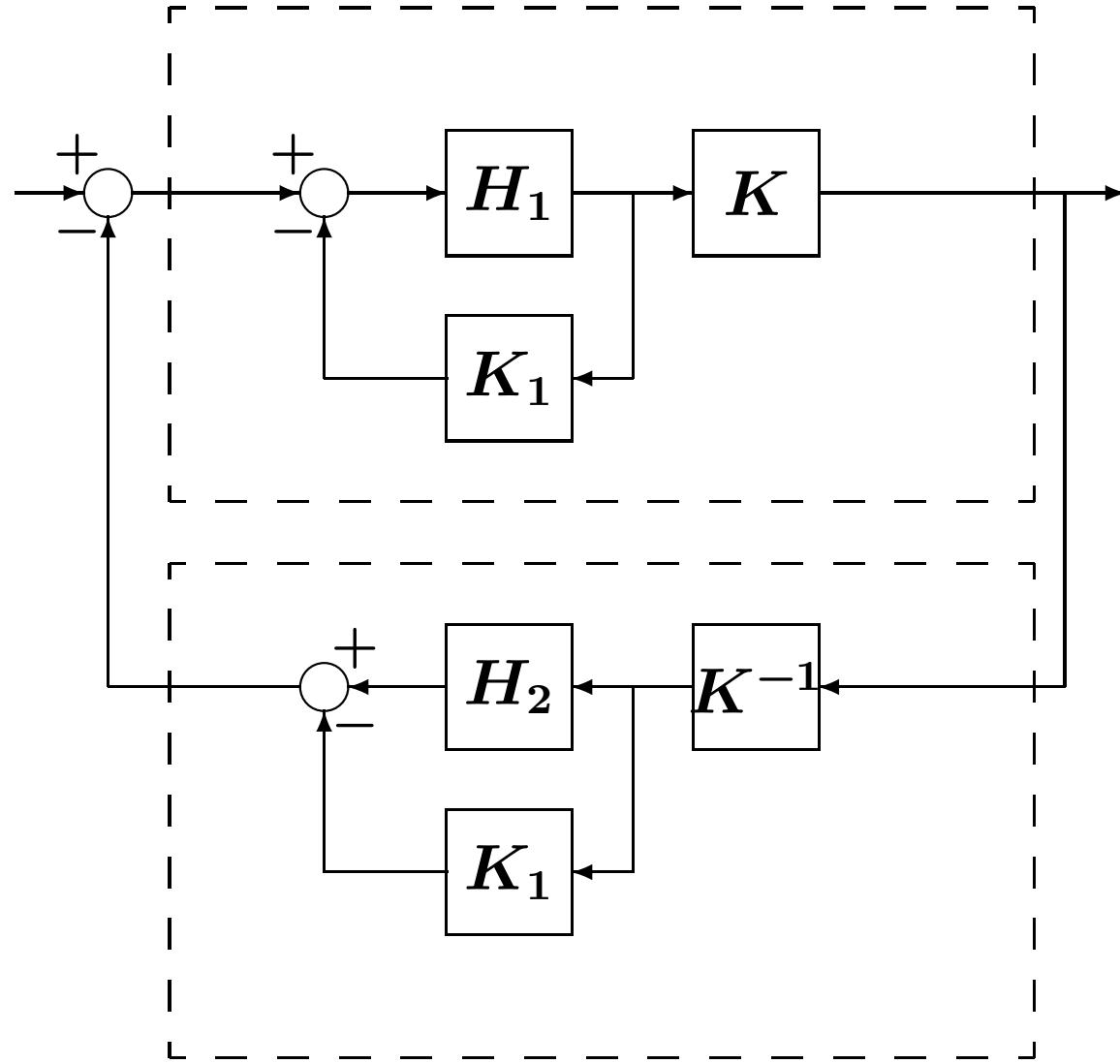


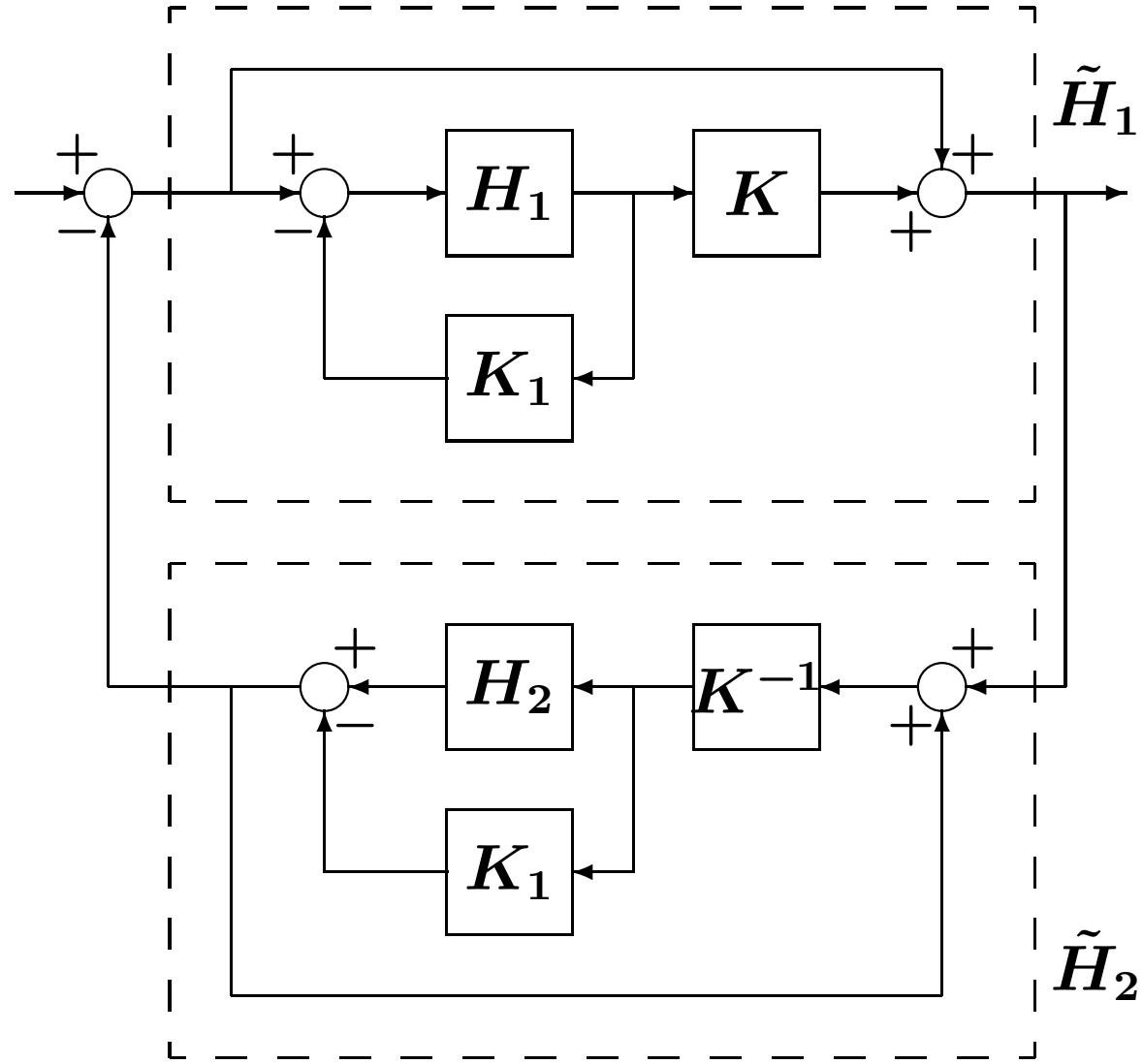


$H_1$  is a dynamical system

$H_2$  is a memoryless function in the sector  $[K_1, K_2]$







## Example (Ex. 6.4)

$$\begin{array}{rcl}
 \dot{x}_1 & = & x_2 \\
 \dot{x}_2 & = & -h(x_1) + bx_2 + e_1 \\
 \underbrace{y_1 & = & x_2}_{H_1} & & \left| \begin{array}{l} \\ \\ \end{array} \right. \underbrace{y_2 = \sigma(e_2)}_{H_2}
 \end{array}$$

$$\sigma \in [\alpha, \beta], \quad h \in [\alpha_1, \infty], \quad b > 0, \quad \alpha_1 > 0, \quad k = \beta - \alpha > 0$$

$$\begin{array}{rcl}
 \dot{x}_1 & = & x_2 \\
 \dot{x}_2 & = & -h(x_1) - ax_2 + \tilde{e}_1 \\
 \underbrace{\tilde{y}_1 & = & kx_2 + \tilde{e}_1}_{\tilde{H}_1} & & \left| \begin{array}{l} \\ \\ \end{array} \right. \underbrace{\tilde{y}_2 = \tilde{\sigma}(\tilde{e}_2)}_{\tilde{H}_2}
 \end{array}$$

$$\tilde{\sigma} \in [0, \infty], \quad a = \alpha - b$$

Assume  $a = \alpha - b > 0$  and show that  $\tilde{H}_1$  is strictly passive

$$V_1 = k \int_0^{x_1} h(s) \, ds + x^T P x$$

$$V_1 = k \int_0^{x_1} h(s) \, ds + p_{11}x_1^2 + 2p_{12}x_1x_2 + p_{22}x_2^2$$

$$\begin{aligned}\dot{V} &= kh(x_1)x_2 + 2(p_{11}x_1 + p_{12}x_2)x_2 \\ &\quad 2(p_{12}x_1 + p_{22}x_2)[-h(x_1) - ax_2 + \tilde{e}_1]\end{aligned}$$

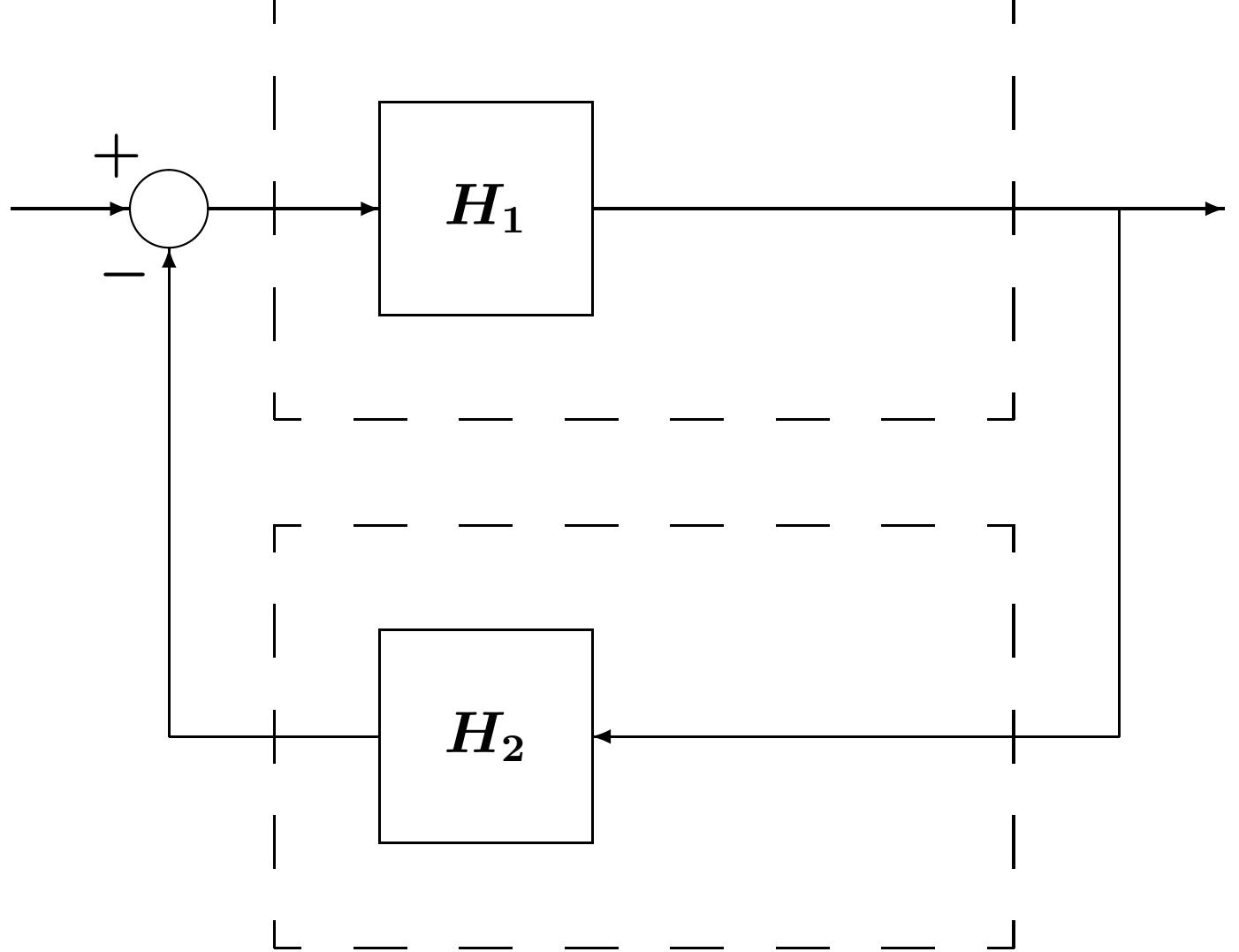
Take  $p_{22} = k/2$ ,  $p_{11} = ap_{12}$

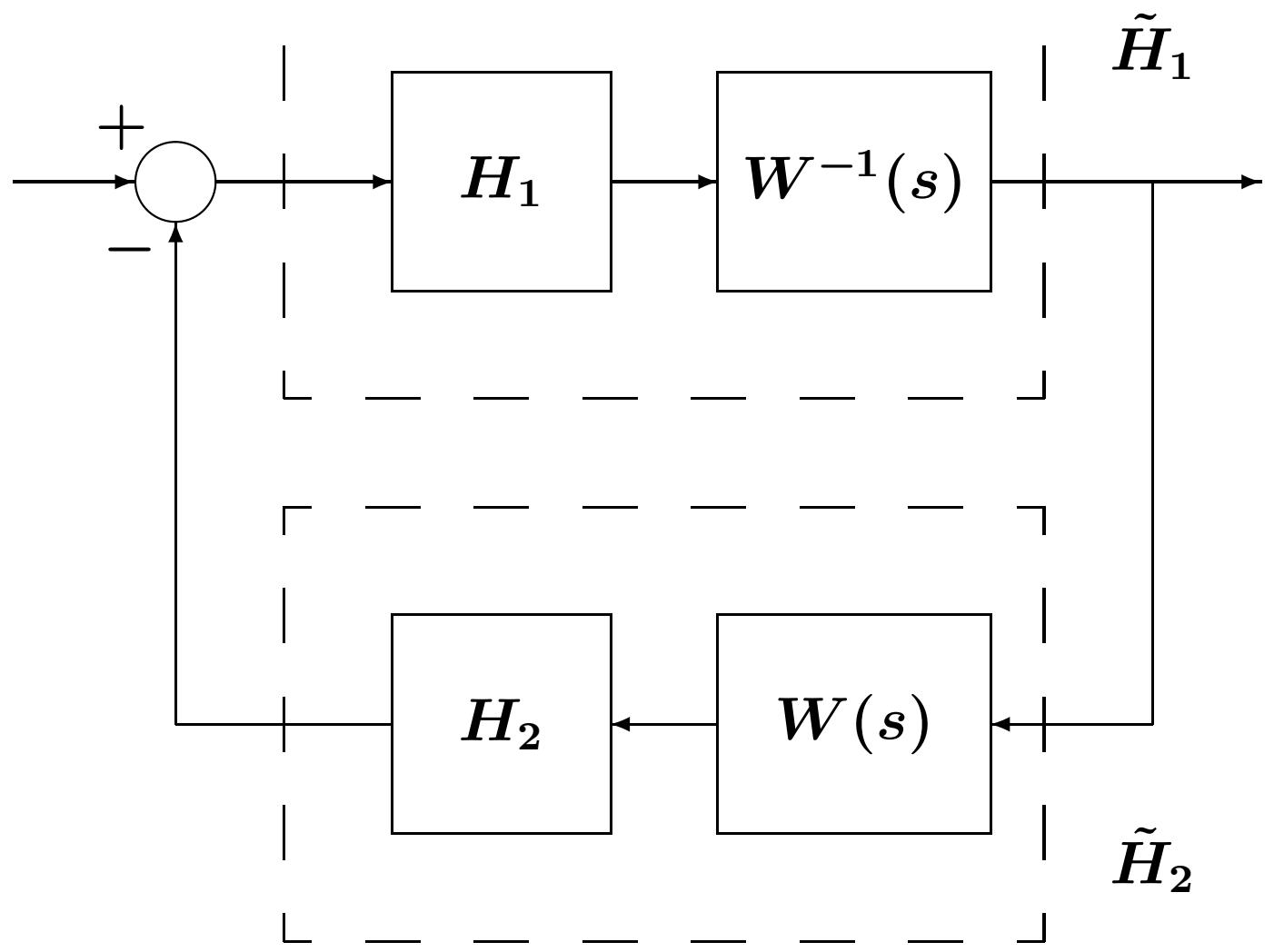
$$\begin{aligned}
\dot{V} &= -2p_{12}x_1h(x_1) - (ka - 2p_{12})x_2^2 \\
&\quad + kx_2\tilde{e}_1 + 2p_{12}x_1\tilde{e}_1 \\
&= -\tilde{e}_1^2 + 2p_{12}x_1\tilde{e}_1 - (ka - 2p_{12})x_2^2 \\
&\quad + \underbrace{(kx_2 + \tilde{e}_1)}_{=\tilde{y}_1}\tilde{e}_1 - 2p_{12}\underbrace{x_1h(x_1)}_{\geq \alpha_1 x_1^2}
\end{aligned}$$

$$\begin{aligned}
\tilde{y}_1\tilde{e}_1 &= \dot{V} + 2p_{12}x_1h(x_1) + (ka - 2p_{12})x_2^2 \\
&\quad + (\tilde{e}_1 - p_{12}x_1)^2 - p_{12}^2x_1^2 \\
&\geq \dot{V} + p_{12}(2\alpha_1 - p_{12})x_1^2 + (ka - 2p_{12})x_2^2
\end{aligned}$$

Take  $0 < p_{12} < \min \left\{ \frac{ak}{2}, 2\alpha_1 \right\}$   $\Rightarrow p_{12}^2 < 2p_{12}\frac{k}{2} = p_{11}p_{22}$

$\tilde{H}_1$  is strictly passive. By Theorem 6.4 the origin is globally asymptotically stable (when  $u = 0$ )





## Example

$$H_1 : \dot{x} = Ax + Be_1, \quad y_1 = Cx$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$H_2 : y_2 = h(e_2), \quad h \in [0, \infty]$$

$$C(sI - A)^{-1}B = \frac{1}{(s^2 + s + 1)} \quad \text{Not PR}$$

$$W(s) = \frac{1}{as + 1} \Rightarrow \frac{(as + 1)}{(s^2 + s + 1)}$$

$$\tilde{H}_1 : \dot{x} = Ax + Be_1, \quad \tilde{y}_1 = \tilde{C}x = \begin{bmatrix} 1 & a \end{bmatrix} x$$

$$\frac{(as + 1)}{(s^2 + s + 1)}$$

$$Re \left[ \frac{1 + j\omega a}{1 - \omega^2 + j\omega} \right] = \frac{1 + (a - 1)\omega^2}{(1 - \omega^2)^2 + \omega^2}$$

$$\lim_{\omega \rightarrow \infty} \omega^2 Re \left[ \frac{1 + j\omega a}{1 - \omega^2 + j\omega} \right] = a - 1$$

$$a > 1 \Rightarrow \frac{(as + 1)}{(s^2 + s + 1)} \text{ is SPR}$$

$$V_1 = \frac{1}{2}x^T Px, \quad PA + A^T P = -L^T L - \varepsilon P, \quad PB = \tilde{C}^T$$

$$\tilde{H}_2 : \quad a\dot{e}_2 = -e_2 + \tilde{e}_2, \quad y_2 = h(e_2), \quad h \in [0, \infty]$$

$\tilde{H}_2$  is strictly passive with  $V_2 = a \int_0^{e_2} h(s) \, ds$ . Use

$$V = V_1 + V_2 = \frac{1}{2}x^T Px + a \int_0^{e_2} h(s) \, ds$$

as a Lyapunov function candidate for the original feedback connection

$$\begin{aligned}
\dot{V} &= \frac{1}{2}x^T P \dot{x} + \frac{1}{2}\dot{x}^T P x + ah(e_2)\dot{e}_2 \\
&= \frac{1}{2}x^T P[Ax - Bh(e_2)] + \frac{1}{2}[Ax - Bh(e_2)]^T Px \\
&\quad + ah(e_2)C[Ax - Bh(e_2)] \\
&= -\frac{1}{2}x^T L^T Lx - (\varepsilon/2)x^T Px - x^T \tilde{C}^T h(e_2) \\
&\quad + ah(e_2)CAx \\
&= -\frac{1}{2}x^T L^T Lx - (\varepsilon/2)x^T Px \\
&\quad - x^T [C + aCA]^T h(e_2) + ah(e_2)CAx \\
&= -\frac{1}{2}x^T L^T Lx - (\varepsilon/2)x^T Px - \underbrace{e_2^T h(e_2)}_{\geq 0} \\
&\leq -(\varepsilon/2)x^T Px
\end{aligned}$$

The origin is globally asymptotically stable