Nonlinear Systems and Control
Lecture # 15
Positive Real Transfer Functions
&
Connection with Lyapunov Stability

Definition: A $p \times p$ proper rational transfer function matrix $G(s)$ is **positive real** if

- poles of all elements of $G(s)$ are in $Re[s] \leq 0$
- for all real ω for which $j\omega$ is not a pole of any element of $G(s)$, the matrix $G(j\omega) + G^T(-j\omega)$ is positive semidefinite
- any pure imaginary pole $j\omega$ of any element of $G(s)$ is a simple pole and the residue matrix $\lim_{s \to j\omega}(s - j\omega)G(s)$ is positive semidefinite Hermitian

$G(s)$ is called **strictly positive real** if $G(s - \varepsilon)$ is positive real for some $\varepsilon > 0$

Scalar Case ($p = 1$):

$$G(j\omega) + G^T(-j\omega) = 2Re[G(j\omega)]$$

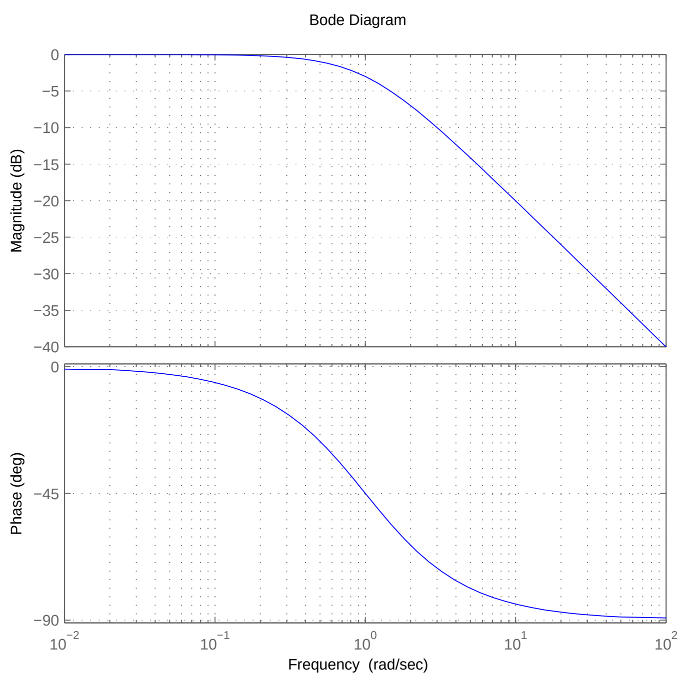
$Re[G(j\omega)]$ is an even function of ω.
The second condition of the definition reduces to

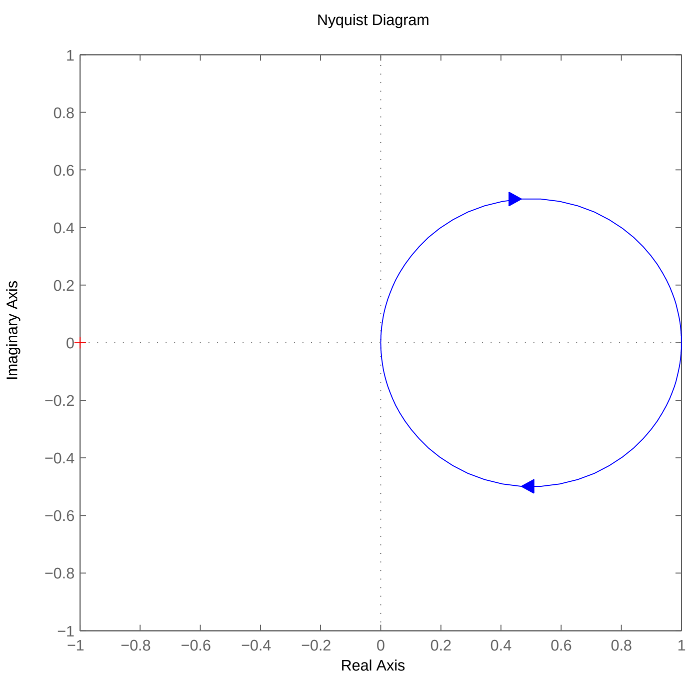
$$Re[G(j\omega)] \geq 0, \ \forall \ \omega \in [0, \infty)$$

which holds when the Nyquist plot of of $G(j\omega)$ lies in the closed right-half complex plane

This is true only if the relative degree of the transfer function is zero or one
Note: for $G(s) = \frac{n(s)}{d(s)}$, the relative degree is deg$d$-deg$n$.

$$G(j\omega) = \frac{1}{j\omega + 1}$$

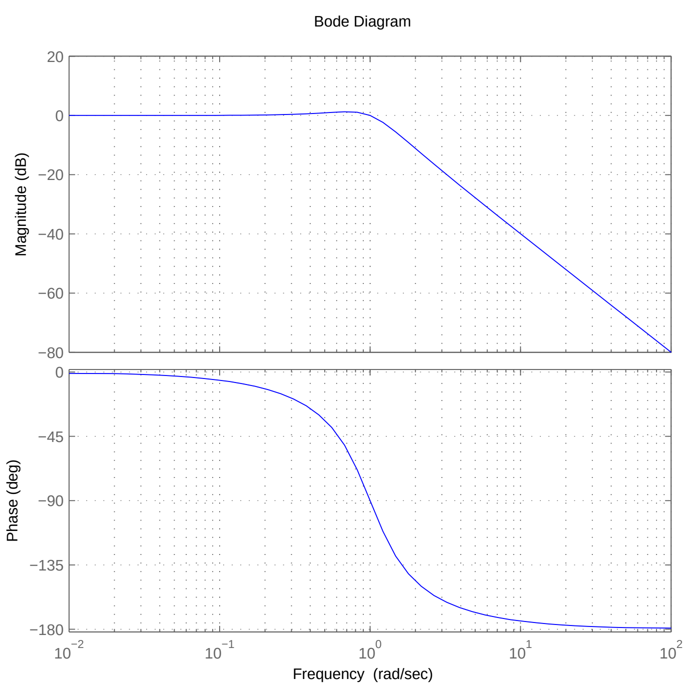


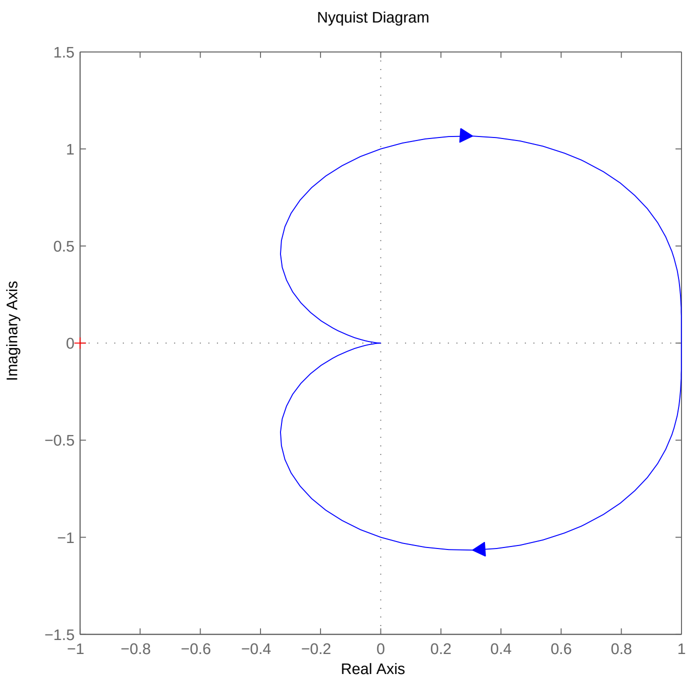


Bode plot

Nyquist plot

$$G(j\omega) = \frac{1}{(j\omega)^2 + j\omega + 1}$$





Bode plot

Nyquist plot

Lemma: Suppose $\det\,[G(s)+G^T(-s)]$ is not identically zero. Then, $G(s)$ is strictly positive real if and only if

- $G(s)$ is Hurwitz
- $G(j\omega)+G^T(-j\omega)>0,\ \forall\,\omega\in R$
- $G(\infty)+G^T(\infty)>0$ or it is positive semidefinte and

$$\lim_{\omega\to\infty}\omega^2 M^T[G(j\omega)+G^T(-j\omega)]M>0$$

for any $p\times(p-q)$ full-rank matrix M such that

$$M^T[G(\infty)+G^T(\infty)]M=0\in R^{(p-q)\times(p-q)}$$

$$q=\mathrm{rank}[G(\infty)+G^T(\infty)]$$

If $G(\infty) + G^T(\infty)$ is singular, the third condition ensures that $G(j\omega) + G^T(-j\omega)$ has

- q singular values with

$$\lim_{\omega\to\infty} \sigma_i(\omega) > 0$$

- $(p - q)$ singular values with

$$\lim_{\omega\to\infty} \sigma_i(\omega) = 0, \quad \lim_{\omega\to\infty} \omega^2 \sigma_i(\omega) > 0$$

Scalar Case ($p = 1$): $G(s)$ is strictly positive real if and only if

- $G(s)$ is Hurwitz
- $Re[G(j\omega)] > 0, \ \forall \ \omega \in [0, \infty)$
- $G(\infty) > 0$ or $G(\infty) = 0$ and

$$\lim_{\omega \to \infty} \omega^2 Re[G(j\omega)] > 0$$

Example:

$$G(s) = \frac{1}{s}$$

has a simple pole at $s = 0$ whose residue is 1

$$Re[G(j\omega)] = Re\left[\frac{1}{j\omega}\right] = 0, \;\; \forall\, \omega \neq 0$$

Hence, G is positive real. It is not strictly positive real since

$$\frac{1}{(s - \varepsilon)}$$

has a pole in $Re[s] > 0$ for any $\varepsilon > 0$

Example:

$$G(s) = \frac{1}{s+a}, \; a > 0, \quad \text{is Hurwitz}$$

$$Re[G(j\omega)] = \frac{a}{\omega^2 + a^2} > 0, \;\; \forall\, \omega \in [0, \infty)$$

$$\lim_{\omega \to \infty} \omega^2 Re[G(j\omega)] = \lim_{\omega \to \infty} \frac{\omega^2 a}{\omega^2 + a^2} = a > 0 \;\; \Rightarrow \;\; \text{G is SPR}$$

Example:

$$G(s) = \frac{1}{s^2 + s + 1}, \;\; Re[G(j\omega)] = \frac{1 - \omega^2}{(1-\omega^2)^2 + \omega^2}$$

G is not PR

Example:

$$G(s) = \begin{bmatrix} \frac{s+2}{s+1} & \frac{1}{s+2} \\ \\ \frac{-1}{s+2} & \frac{2}{s+1} \end{bmatrix} \quad \text{is Hurwitz}$$

$$G(j\omega) + G^T(-j\omega) = \begin{bmatrix} \frac{2(2+\omega^2)}{1+\omega^2} & \frac{-2j\omega}{4+\omega^2} \\ \\ \frac{2j\omega}{4+\omega^2} & \frac{4}{1+\omega^2} \end{bmatrix} > 0, \quad \forall\, \omega \in R$$

$$G(\infty) + G^T(\infty) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\lim_{\omega \to \infty} \omega^2 M^T [G(j\omega) + G^T(-j\omega)] M = 4 \quad \Rightarrow \quad G \text{ is SPR}$$

Positive Real Lemma: Let

$$\boldsymbol{G(s) = C(sI - A)^{-1}B + D}$$

where $(\boldsymbol{A, B})$ is controllable and $(\boldsymbol{A, C})$ is observable. $\boldsymbol{G(s)}$ is positive real if and only if there exist matrices $\boldsymbol{P = P^T > 0}$, $\boldsymbol{L}$, and $\boldsymbol{W}$ such that

$$\begin{aligned} \boldsymbol{PA + A^T P} &= \boldsymbol{-L^T L} \\ \boldsymbol{PB} &= \boldsymbol{C^T - L^T W} \\ \boldsymbol{W^T W} &= \boldsymbol{D + D^T} \end{aligned}$$

Kalman–Yakubovich–Popov Lemma: Let

$$G(s) = C(sI - A)^{-1}B + D$$

where (A, B) is controllable and (A, C) is observable. $G(s)$ is strictly positive real if and only if there exist matrices $P = P^T > 0$, L, and W, and a positive constant ε such that

$$\begin{aligned} PA + A^T P &= -L^T L - \varepsilon P \\ PB &= C^T - L^T W \\ W^T W &= D + D^T \end{aligned}$$

Lemma: The linear time-invariant minimal realization

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

with

$$G(s) = C(sI - A)^{-1}B + D$$

is

- passive if $G(s)$ is positive real
- strictly passive if $G(s)$ is strictly positive real

Proof: Apply the PR and KYP Lemmas, respectively, and use $V(x) = \frac{1}{2}x^T P x$ as the storage function

$$
\begin{aligned}
u^T y - \frac{\partial V}{\partial x} f(x,u) = u^T y - \frac{\partial V}{\partial x}(Ax + Bu) \\
= \; & u^T(Cx + Du) - x^T P(Ax + Bu) \\
= \; & u^T Cx + \tfrac{1}{2} u^T (D + D^T) u \\
& - \tfrac{1}{2} x^T (PA + A^T P) x - x^T PBu \\
= \; & u^T (B^T P + W^T L) x + \tfrac{1}{2} u^T W^T W u \\
& + \tfrac{1}{2} x^T L^T L x + \tfrac{1}{2} \varepsilon x^T P x - x^T PBu \\
= \; & \tfrac{1}{2}(Lx + Wu)^T (Lx + Wu) + \tfrac{1}{2}\varepsilon x^T P x \geq \tfrac{1}{2}\varepsilon x^T P x
\end{aligned}
$$

In the case of the PR Lemma, $\varepsilon = 0$, and we conclude that the system is passive; in the case of the KYP Lemma, $\varepsilon > 0$, and we conclude that the system is strictly passive

Connection with Lyapunov Stability

Lemma: If the system

$$\dot{x} = f(x, u), \qquad y = h(x, u)$$

is passive with a positive definite storage function $V(x)$, then the origin of $\dot{x} = f(x, 0)$ is stable

Proof:

$$u^T y \geq \frac{\partial V}{\partial x} f(x, u) \;\; \Rightarrow \;\; \frac{\partial V}{\partial x} f(x, 0) \leq 0$$

Lemma: If the system

$$\dot{x} = f(x, u), \qquad y = h(x, u)$$

is strictly passive, then the origin of $\dot{x} = f(x, 0)$ is asymptotically stable. Furthermore, if the storage function is radially unbounded, the origin will be globally asymptotically stable

Proof: The storage function $V(x)$ is positive definite

$$u^T y \geq \frac{\partial V}{\partial x} f(x, u) + \psi(x) \;\; \Rightarrow \;\; \frac{\partial V}{\partial x} f(x, 0) \leq -\psi(x)$$

Why is $V(x)$ positive definite?
Let $\phi(t; x)$ be the solution of $\dot{z} = f(z, 0), \; z(0) = x$

$$\dot{V} \le -\psi(x)$$

$$V(\phi(\tau, x)) - V(x) \le -\int_0^{\tau} \psi(\phi(t; x))\, dt, \quad \forall\, \tau \in [0, \delta]$$

$$V(\phi(\tau, x)) \ge 0 \;\Rightarrow\; V(x) \ge \int_0^{\tau} \psi(\phi(t; x))\, dt$$

$$V(\bar{x}) = 0 \;\Rightarrow\; \int_0^{\tau} \psi(\phi(t; \bar{x}))\, dt = 0, \; \forall\, \tau \in [0, \delta]$$

$$\Rightarrow\; \psi(\phi(t; \bar{x})) \equiv 0 \;\Rightarrow\; \phi(t; \bar{x}) \equiv 0 \;\Rightarrow\; \bar{x} = 0$$

Definition: The system

$$\dot{x} = f(x, u), \qquad y = h(x, u)$$

is zero-state observable if no solution of $\dot{x} = f(x, 0)$ can stay identically in $S = \{h(x, 0) = 0\}$, other than the zero solution $x(t) \equiv 0$

Linear Systems

$$\dot{x} = Ax, \quad y = Cx$$

Observability of (A, C) is equivalent to

$$y(t) = Ce^{At}x(0) \equiv 0 \;\Leftrightarrow\; x(0) = 0 \;\Leftrightarrow\; x(t) \equiv 0$$

If (A, C) is observable

$$y(t) = Ce^{At}x(0) \equiv 0 \;\Leftrightarrow\; x(0) = 0$$

Proof: $(\Leftarrow)$ trivial

$(\Rightarrow)$ Suppose not, i.e., $y(t) = Ce^{At}x(0) \equiv 0 \Rightarrow x(0) \neq 0$

$$\text{Cayley Hamilton} \quad \left[\sum_{k=0}^{n-1} \alpha_k(t) C A^k\right] x(0) \equiv 0$$

$$\Leftrightarrow \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x(0) \equiv 0$$

Lemma: If the system

$$\dot{x} = f(x,u), \qquad y = h(x,u)$$

is output strictly passive and zero-state observable, then the origin of $\dot{x} = f(x,0)$ is asymptotically stable. Furthermore, if the storage function is radially unbounded, the origin will be globally asymptotically stable

Proof: The storage function $V(x)$ is positive definite

$$u^T y \geq \frac{\partial V}{\partial x} f(x,u) + y^T \rho(y) \;\; \Rightarrow \;\; \frac{\partial V}{\partial x} f(x,0) \leq -y^T \rho(y)$$

$$\dot{V}(x(t)) \equiv 0 \; \Rightarrow \; y(t) \equiv 0 \Rightarrow x(t) \equiv 0$$

Apply the invariance principle

Example

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -ax_1^3 - kx_2 + u, \quad y = x_2, \quad a, k > 0$$

$$V(x) = \tfrac{1}{4}ax_1^4 + \tfrac{1}{2}x_2^2$$

$$\dot{V} = ax_1^3 x_2 + x_2(-ax_1^3 - kx_2 + u) = -ky^2 + yu$$

The system is output strictly passive

$$y(t) \equiv 0 \;\Leftrightarrow\; x_2(t) \equiv 0 \;\Rightarrow\; ax_1^3(t) \equiv 0 \;\Rightarrow\; x_1(t) \equiv 0$$

The system is zero-state observable. V is radially unbounded. Hence, the origin of the unforced system is globally asymptotically stable