Nonlinear Systems and Control
Lecture # 13
Perturbed Systems

Nominal System:

$$\dot{x} = f(x), \qquad f(0) = 0$$

Perturbed System:

$$\dot{x} = f(x) + g(t, x), \qquad g(t, 0) = 0$$

Case 1: The origin of the nominal system is exponentially stable

$$c_1\|x\|^2 \le V(x) \le c_2\|x\|^2$$

$$\frac{\partial V}{\partial x} f(x) \le -c_3\|x\|^2$$

$$\left\| \frac{\partial V}{\partial x} \right\| \le c_4\|x\|$$

Use $V(x)$ as a Lyapunov function candidate for the perturbed system

$$\dot{V}(t,x) = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(t,x)$$

Assume that

$$\|g(t,x)\| \le \gamma \|x\|, \quad \gamma \ge 0$$

$$\begin{aligned} \dot{V}(t,x) &\le -c_3\|x\|^2 + \left\|\frac{\partial V}{\partial x}\right\| \|g(t,x)\| \\ &\le -c_3\|x\|^2 + c_4\gamma\|x\|^2 \end{aligned}$$

$$\gamma < \frac{c_3}{c_4}$$

$$\dot{V}(t,x) \leq -(c_3 - \gamma c_4)\|x\|^2$$

The origin is an exponentially stable equilibrium point of the perturbed system

Example

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -4x_1 - 2x_2 + \beta x_2^3, \qquad \beta \geq 0
\end{aligned}
$$

$$\dot{x} = Ax + g(x)$$

$$A = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix}, \qquad g(x) = \begin{bmatrix} 0 \\ \beta x_2^3 \end{bmatrix}$$

The eigenvalues of A are $-1 \pm j\sqrt{3}$

$$PA + A^T P = -I \quad \Rightarrow \quad P = \begin{bmatrix} \frac{3}{2} & \frac{1}{8} \\ \frac{1}{8} & \frac{5}{16} \end{bmatrix}$$

For the quadratic Lyapunv function $\boldsymbol{V(x) = x^T P x}$,

$$\underbrace{\boldsymbol{\lambda_{\min}(P)}}_{=:c_1} \|x\|_2^2 \leq V(x) \leq \underbrace{\boldsymbol{\lambda_{\max}(P)}}_{=:c_2} \|x\|_2^2$$

$$\mathcal{L}_{Ax} V = \frac{\partial V}{\partial x} A x = -x^T Q x \leq -\underbrace{\lambda_{\min}(Q)}_{=:c_3} \|x\|_2^2$$

$$\left\| \frac{\partial V}{\partial x} \right\|_2 = \|2x^T P\| \leq 2\|P\|_2 \|x\|_2 = \underbrace{2\lambda_{\max}(P)}_{=:c_4} \|x\|_2$$

$$V(x) = x^T P x, \quad \frac{\partial V}{\partial x} A x = -x^T x$$

$$c_3 = 1, \;\; c_4 = 2\,\|P\| = 2\lambda_{\max}(P) = 2 \times 1.513 = 3.026$$

$$\|g(x)\| = \beta |x_2|^3 \le \beta k_2^2 |x_2| \le \beta k_2^2 \|x\|, \; \forall |x_2| \le k_2$$

$g(x)$ satisfies the bound $\|g(x)\| \le \gamma \|x\|$ over compact sets of x. Consider the compact set

$$\Omega_c = \{V(x) \le c\} = \{x^T P x \le c\}, \quad c > 0$$

$$k_2 = \max_{x^T P x \le c} |x_2| = \max_{x^T P x \le c} |[0 \;\; 1] x|$$

Fact:

$$\max_{x^T P x \le c} \|Lx\| = \sqrt{c}\,\|LP^{-1/2}\|$$

Proof

$$x^T P x \le c \;\Leftrightarrow\; \frac{1}{c} x^T P x \le 1 \;\Leftrightarrow\; \frac{1}{c}\, x^T P^{1/2}\, P^{1/2} x \le 1$$

$$y = \frac{1}{\sqrt{c}}\, P^{1/2} x$$

$$\max_{x^T P x \le c} \|Lx\| = \max_{y^T y \le 1} \|L\sqrt{c}\, P^{-1/2} y\| = \sqrt{c}\,\|LP^{-1/2}\|$$

$$k_2 = \max_{x^T P x \le c} |[0 \;\; 1]x| = \sqrt{c}\, \|[0 \;\; 1]P^{-1/2}\| = 1.8194\sqrt{c}$$

$$\|g(x)\| \le \beta\, c\, (1.8194)^2 \|x\|, \quad \forall\, x \in \Omega_c$$

$$\|g(x)\| \le \gamma \|x\|, \quad \forall\, x \in \Omega_c, \quad \gamma = \beta\, c\, (1.8194)^2$$

$$\gamma < \frac{c_3}{c_4} \;\; \Leftrightarrow \;\; \beta < \frac{1}{3.026 \times (1.8194)^2 c} \approx \frac{0.1}{c}$$

$$\beta < 0.1/c \;\; \Rightarrow \;\; \dot{V}(x) \le -(1 - 10\beta c)\|x\|^2$$

Hence, the origin is exponentially stable and Ω_c is an estimate of the region of attraction

Alternative Bound on β

$$\begin{aligned}\dot{V}(x) &= -\|x\|^2 + 2x^T P g(x) \\ &= -\|x\|^2 + \tfrac{1}{8}\beta x_2^3 \left([2 \;\; 5]x\right) \\ &\le -\|x\|^2 + \tfrac{\sqrt{29}}{8}\beta x_2^2 \|x\|^2\end{aligned}$$

Over Ω_c, $x_2^2 \le (1.8194)^2 c$

$$\begin{aligned}\dot{V}(x) &\le -\left(1 - \tfrac{\sqrt{29}}{8}\beta (1.8194)^2 c\right)\|x\|^2 \\ &= -\left(1 - \frac{\beta c}{0.448}\right)\|x\|^2\end{aligned}$$

If $\beta < 0.448/c$, the origin will be exponentially stable and Ω_c will be an estimate of the region of attraction

Remark: The inequality $\boldsymbol{\beta < 0.448/c}$ shows a tradeoff between the estimate of the region of attraction and the estimate of the upper bound on $\boldsymbol{\beta}$

The smaller the upper bound on $\boldsymbol{\beta}$, the larger the estimate of RA

Case 2: The origin of the nominal system is asymptotically stable

$$\dot{V}(t,x) = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(t,x) \leq -W_3(x) + \left\| \frac{\partial V}{\partial x} g(t,x) \right\|$$

Under what condition will the following inequality hold?

$$\left\| \frac{\partial V}{\partial x} g(t,x) \right\| < W_3(x)$$

Special Case: Quadratic-Type Lyapunov function

$$\frac{\partial V}{\partial x} f(x) \leq -c_3 \phi^2(x), \quad \left\| \frac{\partial V}{\partial x} \right\| \leq c_4 \phi(x)$$

$\phi(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ is positive definite and continuous

$$\dot{V}(t,x) \leq -c_3\phi^2(x) + c_4\phi(x)\|g(t,x)\|$$

$$\text{If } \; \|g(t,x)\| \leq \gamma\phi(x), \quad \text{with } \gamma < \frac{c_3}{c_4}$$

$$\dot{V}(t,x) \leq -(c_3 - c_4\gamma)\phi^2(x)$$

Example

$$\dot{x} = -x^3 + g(t,x)$$

$V(x) = x^4$ is a quadratic-type Lyapunov function for the nominal system $\dot{x} = -x^3$

$$\frac{\partial V}{\partial x}(-x^3) = -4x^6, \qquad \left|\frac{\partial V}{\partial x}\right| = 4|x|^3$$

$$\phi(x) = |x|^3, \quad c_3 = 4, \quad c_4 = 4$$

Suppose $|g(t,x)| \le \gamma |x|^3, \ \forall \ x,$ with $\gamma < 1$

$$\dot{V}(t,x) \le -4(1-\gamma)\phi^2(x)$$

Hence, the origin is a globally uniformly asymptotically stable

Remark: A nominal system with asymptotically, but not exponentially, stable origin is not robust to smooth perturbations with arbitrarily small linear growth bounds

Example

$$\dot{x} = -x^3 + \gamma x$$

The origin is unstable for any $\gamma > 0$ (can be easily seen via linearization)