Nonlinear Systems and Control
Lecture # 12
Converse Lyapunov Functions
&
Time Varying Systems

Converse Lyapunov Theorem–Exponential Stability

Let $x = 0$ be an exponentially stable equilibrium point for the system $\dot{x} = f(x)$, where f is continuously differentiable on $D = \{\|x\| < r\}$. Let k, λ, and r_0 be positive constants with $r_0 < r/k$ such that

$$\|x(t)\| \leq k\|x(0)\|e^{-\lambda t}, \quad \forall\, x(0) \in D_0,\ \forall\, t \geq 0$$

where $D_0 = \{\|x\| < r_0\}$. Then, there is a continuously differentiable function $V(x)$ that satisfies the inequalities

$$c_1\|x\|^2 \le V(x) \le c_2\|x\|^2$$

$$\frac{\partial V}{\partial x} f(x) \le -c_3\|x\|^2$$

$$\left\|\frac{\partial V}{\partial x}\right\| \le c_4\|x\|$$

for all $x \in D_0$, with positive constants c_1, c_2, c_3, and c_4 Moreover, if f is continuously differentiable for all x, globally Lipschitz, and the origin is globally exponentially stable, then $V(x)$ is defined and satisfies the aforementioned inequalities for all $x \in R^n$

Idea of the proof: Let $\psi(t;x)$ be the solution of

$$\dot{y} = f(y), \quad y(0) = x$$

Take

$$V(x) = \int_0^\delta \psi^T(t;x)\,\psi(t;x)\,dt, \qquad \delta > 0$$

Application: Consider the system $\dot{x} = f(x)$ where f is continuously differentiable in the neighborhood of the origin and $f(0) = 0$. Show that the origin is exponentially stable only if $A = [\partial f/\partial x](0)$ is Hurwitz

$$f(x) = Ax + G(x)x, \quad G(x) \to 0 \text{ as } x \to 0$$

Given any $L > 0$, there is $r_1 > 0$ such that

$$\|G(x)\| \le L, \ \forall \ \|x\| < r_1$$

Because the origin of $\dot{x} = f(x)$ is exponentially stable, let $V(x)$ be the function provided by the converse Lyapunov theorem over the domain $\{\|x\| < r_0\}$. Use $V(x)$ as a Lyapunov function candidate for $\dot{x} = Ax$

$$
\begin{aligned}
\frac{\partial V}{\partial x} A x &= \frac{\partial V}{\partial x} f(x) - \frac{\partial V}{\partial x} G(x) x \\
&\leq -c_3 \|x\|^2 + c_4 L \|x\|^2 \\
&= -(c_3 - c_4 L)\|x\|^2
\end{aligned}
$$

Take $L < c_3/c_4, \ \ \gamma \stackrel{\text{def}}{=} (c_3 - c_4 L) > 0 \ \Rightarrow$

$$
\frac{\partial V}{\partial x} A x \leq -\gamma \|x\|^2, \ \ \forall \ \|x\| < \min\{r_0, \ r_1\}
$$

The origin of $\dot{x} = Ax$ is exponentially stable

Time-varying Systems

$$\dot{x} = f(t, x)$$

$f(t, x)$ is piecewise continuous in t and locally Lipschitz in x for all $t \geq 0$ and all $x \in D$. The origin is an equilibrium point at $t = 0$ if

$$f(t, 0) = 0, \;\; \forall\, t \geq 0$$

While the solution of the autonomous system

$$\dot{x} = f(x), \quad x(t_0) = x_0$$

depends only on $(t - t_0)$, the solution of

$$\dot{x} = f(t, x), \quad x(t_0) = x_0$$

may depend on both t and t_0

Comparison Functions

- A scalar continuous function $\alpha(r)$, defined for $r \in [0, a)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if it defined for all $r \geq 0$ and $\alpha(r) \longrightarrow \infty$ as $r \longrightarrow \infty$

- A scalar continuous function $\beta(r, s)$, defined for $r \in [0, a)$ and $s \in [0, \infty)$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \longrightarrow 0$ as $s \longrightarrow \infty$

Example

- $\alpha(r) = \tan^{-1}(r)$ is strictly increasing since $\alpha'(r) = 1/(1+r^2) > 0$. It belongs to class $\mathcal{K}$, but not to class $\mathcal{K}_\infty$ since $\lim_{r\to\infty} \alpha(r) = \pi/2 < \infty$

- $\alpha(r) = r^c$, for any positive real number c, is strictly increasing since $\alpha'(r) = cr^{c-1} > 0$. Moreover, $\lim_{r\to\infty} \alpha(r) = \infty$; thus, it belongs to class $\mathcal{K}_\infty$

- $\alpha(r) = \min\{r, r^2\}$ is continuous, strictly increasing, and $\lim_{r\to\infty} \alpha(r) = \infty$. Hence, it belongs to class $\mathcal{K}_\infty$

- $\beta(r,s) = r/(ksr+1)$, for any positive real number k, is strictly increasing in r since

$$\frac{\partial \beta}{\partial r} = \frac{1}{(ksr+1)^2} > 0$$

and strictly decreasing in s since

$$\frac{\partial \beta}{\partial s} = \frac{-kr^2}{(ksr+1)^2} < 0$$

Moreover, $\beta(r,s) \to 0$ as $s \to \infty$. Therefore, it belongs to class $\mathcal{KL}$

- $\beta(r,s) = r^c e^{-s}$, for any positive real number c, belongs to class $\mathcal{KL}$

Properties:

- α belongs to class $\mathcal{K}$ (or $\mathcal{K}_\infty$) $\Rightarrow \alpha^{-1}$ belongs to class $\mathcal{K}$ (or $\mathcal{K}_\infty$)
- α_1, α_2 belong to $\mathcal{K}$ (or $\mathcal{K}_\infty$) $\Rightarrow \alpha_1 \circ \alpha_2$ belongs to class $\mathcal{K}$ (or $\mathcal{K}_\infty$)
- W be a continuous positive definite function on D that contains the origin. Let $Br \subset D$, then there exist class $\mathcal{K}$ functions α_1, α_2 on $[0, r]$, such that

$$\alpha_1(\|x\|) \leq W(x) \leq \alpha_2(\|x\|)$$

for all $x \in B_r$. If $W(x)$ is radially unbounded, then α_1 and α_2 can be chosen to belong to class $\mathcal{K}_\infty$

Definition: The equilibrium point $x = 0$ of $\dot{x} = f(t, x)$ is

- uniformly stable if there exist a class $\mathcal{K}$ function α and a positive constant c, independent of t_0, such that

$$\|x(t)\| \le \alpha(\|x(t_0)\|), \ \forall\, t \ge t_0 \ge 0, \ \forall\, \|x(t_0)\| < c$$

- uniformly asymptotically stable if there exist a class $\mathcal{KL}$ function β and a positive constant c, independent of t_0, such that

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0), \ \forall\, t \ge t_0 \ge 0, \ \forall\, \|x(t_0)\| < c$$

- globally uniformly asymptotically stable if the foregoing inequality is satisfied for any initial state $x(t_0)$

- exponentially stable if there exist positive constants c, k, and λ such that

$$\|x(t)\| \le k\|x(t_0)\| e^{-\lambda(t-t_0)}, \ \forall \ \|x(t_0)\| < c$$

- globally exponentially stable if the foregoing inequality is satisfied for any initial state $x(t_0)$

Theorem: Let the origin $x = 0$ be an equilibrium point for $\dot{x} = f(t, x)$ and $D \subset R^n$ be a domain containing $x = 0$. Suppose $f(t, x)$ is piecewise continuous in t and locally Lipschitz in x for all $t \geq 0$ and $x \in D$. Let $V(t, x)$ be a continuously differentiable function such that

$$W_1(x) \leq V(t, x) \leq W_2(x) \quad (1)$$

$$\dot{V}(t, x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq 0 \quad (2)$$

for all $t \geq 0$ and $x \in D$, where $W_1(x)$ and $W_2(x)$ are continuous positive definite functions on D. Then, the origin is uniformly stable

Proof: Choose $B_r \subset D$ and $c < \min_{\|x\|=r} W_1(x)$.
Take $\Omega_{t,c} = \{x \in B_r | V(t,x) \le c\}$.
$\dot{V}$ in $\Omega_{t,c} \Rightarrow \Omega_{t,c}$ is positively invariant. $\Omega_{t,c}$ depends on t.
Sandwich $\Omega_{t,c}$ between two sets, which are indep. of t.
$\{x \in B_r | W_2(x) \le c\}$

$$W_2(x) \le c \Rightarrow V(t,x) \le c \Rightarrow \{W_2(x) \le c\} \subset \{V(t,x) \le c\}$$

$\{x \in B_r | W_1(x) \le c\}$

$$V(t,x) \le c \Rightarrow W_1(x) \le c \Rightarrow \{V(t,x) \le c\} \subset \{W_1(x) \le c\}$$

Hence, the solution is bounded and defined for all $t \ge t_0$.
Moreover, for all $t \ge t_0$,

$$V(t, x(t)) \le V(t_0, x(t_0))$$

$\exists$ class $\mathcal{K}$ functions α_1 and α_2 such that

$$\alpha_1(\|x\|) \le W_1(x) \le V(t,x) \le W_2(x) \le \alpha_2(\|x\|)$$

$$\begin{aligned} \|x(t)\| &\le \alpha_1^{-1}(V(t,x(t))) \le \alpha_1^{-1}(V(t_0,x(t_0))) \\ &\le \alpha_1^{-1}(\alpha_2(\|x(t_0)\|)) \end{aligned}$$

$\alpha_1^{-1} \circ \alpha_2$ is a class $\mathcal{K}$ function, hence the origin is uniformly stable

Theorem: Suppose the assumptions of the previous theorem are satisfied with

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \le -W_3(x)$$

for all $t \ge 0$ and $x \in D$, where $W_3(x)$ is a continuous positive definite function on D. Then, the origin is uniformly asymptotically stable. Moreover, if r and c are chosen such that $B_r = \{\|x\| \le r\} \subset D$ and $c < \min_{\|x\|=r} W_1(x)$, then every trajectory starting in $\{x \in B_r \mid W_2(x) \le c\}$ satisfies

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0), \quad \forall\, t \ge t_0 \ge 0$$

for some class $\mathcal{KL}$ function β. Finally, if $D = R^n$ and $W_1(x)$ is radially unbounded, then the origin is globally uniformly asymptotically stable

Theorem: Suppose the assumptions of the previous theorem are satisfied with

$$k_1\|x\|^a \le V(t,x) \le k_2\|x\|^a$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le -k_3\|x\|^a$$

for all $t \ge 0$ and $x \in D$, where k_1, k_2, k_3, and a are positive constants. Then, the origin is exponentially stable. If the assumptions hold globally, the origin will be globally exponentially stable.

Example:

$$\dot{x} = -[1 + g(t)]x^3, \quad g(t) \geq 0, \ \forall \ t \geq 0$$

$$V(x) = \tfrac{1}{2}x^2$$

$$\dot{V}(t,x) = -[1 + g(t)]x^4 \leq -x^4, \quad \forall \, x \in R, \ \forall \, t \geq 0$$

The origin is globally uniformly asymptotically stable

Example:

$$\begin{aligned} \dot{x}_1 &= -x_1 - g(t)x_2 \\ \dot{x}_2 &= x_1 - x_2 \end{aligned}$$

$$0 \leq g(t) \leq k \ \text{ and } \ \dot{g}(t) \leq g(t), \ \forall \, t \geq 0$$

$$V(t,x) = x_1^2 + [1+g(t)]x_2^2$$

$$x_1^2 + x_2^2 \le V(t,x) \le x_1^2 + (1+k)x_2^2, \quad \forall\, x \in R^2$$

$$\dot{V}(t,x) = -2x_1^2 + 2x_1x_2 - [2 + 2g(t) - \dot{g}(t)]x_2^2$$

$$2 + 2g(t) - \dot{g}(t) \ge 2 + 2g(t) - g(t) \ge 2$$

$$\dot{V}(t,x) \le -2x_1^2 + 2x_1x_2 - 2x_2^2 = -\,x^T \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} x$$

The origin is globally exponentially stable