Nonlinear Systems and Control Lecture # 10
The Invariance Principle

Example: Pendulum equation with friction

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a\sin x_1 - bx_2$$

$$V(x) = a(1 - \cos x_1) + \frac{1}{2}x_2^2$$

$$\dot{V}(x) = a\dot{x}_1 \sin x_1 + x_2\dot{x}_2 = -bx_2^2$$

The origin is stable. $\dot{V}(x)$ is not negative definite because $\dot{V}(x) = 0$ for $x_2 = 0$ irrespective of the value of x_1

However, near the origin $x_1 \neq 0$, the solution cannot stay identically in the set $\{x_2 = 0\}$

Definitions: Let $x(t)$ be a solution of $\dot{x} = f(x)$

A point p is said to be a *positive limit point* of $x(t)$ if there is a sequence $\{t_n\}$, with $\lim_{n \to \infty} t_n = \infty$, such that $x(t_n) \to p$ as $n \to \infty$

The set of all positive limit points of $x(t)$ is called the *positive limit set* of $x(t)$; denoted by L^+

If $x(t)$ approaches an asymptotically stable equilibrium point $\bar{x}$, then $\bar{x}$ is the positive limit point of $x(t)$ and $L^+ = \bar{x}$

A stable limit cycle is the positive limit set of every solution starting sufficiently near the limit cycle

A set M is an *invariant set* with respect to $\dot{x} = f(x)$ if

$$x(0) \in M \Rightarrow x(t) \in M, \quad \forall \, t \in R$$

Examples:

- Equilibrium points

- Limit Cycles

A set M is a *positively invariant set* with respect to $\dot{x} = f(x)$ if

$$x(0) \in M \Rightarrow x(t) \in M, \quad \forall \, t \geq 0$$

Example:

The set $\Omega_c = \{V(x) \leq c\}$ with $\dot{V}(x) \leq 0$ in Ω_c

The distance from a point p to a set M is defined by

$$\text{dist}(p, M) = \inf_{x \in M} \|p - x\|$$

$x(t)$ approaches a set M as t approaches infinity, if for each $\varepsilon > 0$ there is $T > 0$ such that

$$\text{dist}(x(t), M) < \varepsilon, \quad \forall\, t > T$$

Example: every solution $x(t)$ starting sufficiently near a stable limit cycle approaches the limit cycle as $t \to \infty$

Notice, however, that $x(t)$ does not converge to any specific point on the limit cycle

Lemma: If a solution $x(t)$ of $\dot{x} = f(x)$ is bounded and belongs to D for $t \geq 0$, then its positive limit set L^+ is a nonempty, compact, invariant set. Moreover, $x(t)$ approaches L^+ as $t \to \infty$

LaSalle's theorem: Let $f(x)$ be a locally Lipschitz function defined over a domain $D \subset R^n$ and $\Omega \subset D$ be a compact set that is positively invariant with respect to $\dot{x} = f(x)$. Let $V(x)$ be a continuously differentiable function defined over D such that $\dot{V}(x) \leq 0$ in Ω. Let E be the set of all points in Ω where $\dot{V}(x) = 0$, and M be the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$

Note:

- Here "largest" invariant set is understood in the sense of set theory, i.e., M is the union of all invariant sets (such as equilibrium points or limit cycles) within E.

- If once $\dot{V}(x) = 0$, then $\dot{V}(x) \equiv 0$ for all future time, then $M = E$.

- Here $V(x)$ does not have to be positive definite.

Proof:

$$\dot{V}(x) \leq 0 \ \text{in} \ \Omega(\text{compact}) \ \Rightarrow \ V(x(t)) \ \text{is a decreasing}$$

$$V(x) \ \text{is continuous in} \ \Omega \ \Rightarrow \ V(x) \geq b = \min_{x \in \Omega} V(x)$$

$$\Rightarrow \ \lim_{t \to \infty} V(x(t)) = a$$

$$x(t) \in \Omega \ \Rightarrow \ x(t) \ \text{is bounded} \ \Rightarrow \ L^+ \ \text{exists (Lemma)}$$

Moreover, $L^+ \subset \Omega$ and $x(t)$ approaches L^+ as $t \to \infty$

For any $p \in L^+$, there is $\{t_n\}$ with $\lim_{n \to \infty} t_n = \infty$ such that $x(t_n) \to p$ as $n \to \infty$

$$V(x) \ \text{is continuous} \ \Rightarrow \ V(p) = \lim_{n \to \infty} V(x(t_n)) = a$$

Why?

$V(x)$ is continuous $\Rightarrow$ $V(p) = \lim_{n \to \infty} V(x(t_n)) = a$.

$$\forall \delta, \exists N, \text{ s.t. } \|x(t_n) - p\| < \delta, \forall n \geq N,$$

$$\forall \epsilon, \exists \delta, \text{ s.t. } \|x - p\| < \delta \Rightarrow \|V(x) - V(p)\| < \epsilon,$$

$$\Rightarrow \forall \epsilon, \exists N, \text{ s.t. } \|V(x(t_n)) - V(p)\| < \epsilon, \forall n \geq N$$

$$V(x) = a \text{ on } L^+ \text{ and } L^+ \text{ invariant} \Rightarrow \dot{V}(x) = 0, \forall x \in L^+$$

$$L^+ \subset M \subset E \subset \Omega$$

$x(t)$ approaches L^+ $\Rightarrow$ $x(t)$ approaches M (as $t \to \infty$)

Theorem: Let $f(x)$ be a locally Lipschitz function defined over a domain $D \subset R^n$; $0 \in D$. Let $V(x)$ be a continuously differentiable positive definite function defined over D such that $\dot{V}(x) \leq 0$ in D. Let $S = \{x \in D \mid \dot{V}(x) = 0\}$

- If no solution can stay identically in S, other than the trivial solution $x(t) \equiv 0$, then the origin is asymptotically stable

- Moreover, if $\Gamma \subset D$ is compact and positively invariant, then it is a subset of the region of attraction

- Furthermore, if $D = R^n$ and $V(x)$ is radially unbounded, then the origin is globally asymptotically stable

Example:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -h_1(x_1) - h_2(x_2)
\end{aligned}$$

$$h_i(0) = 0, \quad y h_i(y) > 0, \quad \text{for } 0 < |y| < a$$

$$V(x) = \int_0^{x_1} h_1(y) \, dy \; + \; \tfrac{1}{2} x_2^2$$

$$D = \{-a < x_1 < a, \;\; -a < x_2 < a\}$$

$$\dot{V}(x) = h_1(x_1) x_2 + x_2[-h_1(x_1) - h_2(x_2)] = -x_2 h_2(x_2) \le 0$$

$$\dot{V}(x) = 0 \;\Rightarrow\; x_2 h_2(x_2) = 0 \;\Rightarrow\; x_2 = 0$$

$$S = \{x \in D \mid x_2 = 0\}$$

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -h_1(x_1) - h_2(x_2)$$

$$x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0 \Rightarrow h_1(x_1(t)) \equiv 0 \Rightarrow x_1(t) \equiv 0$$

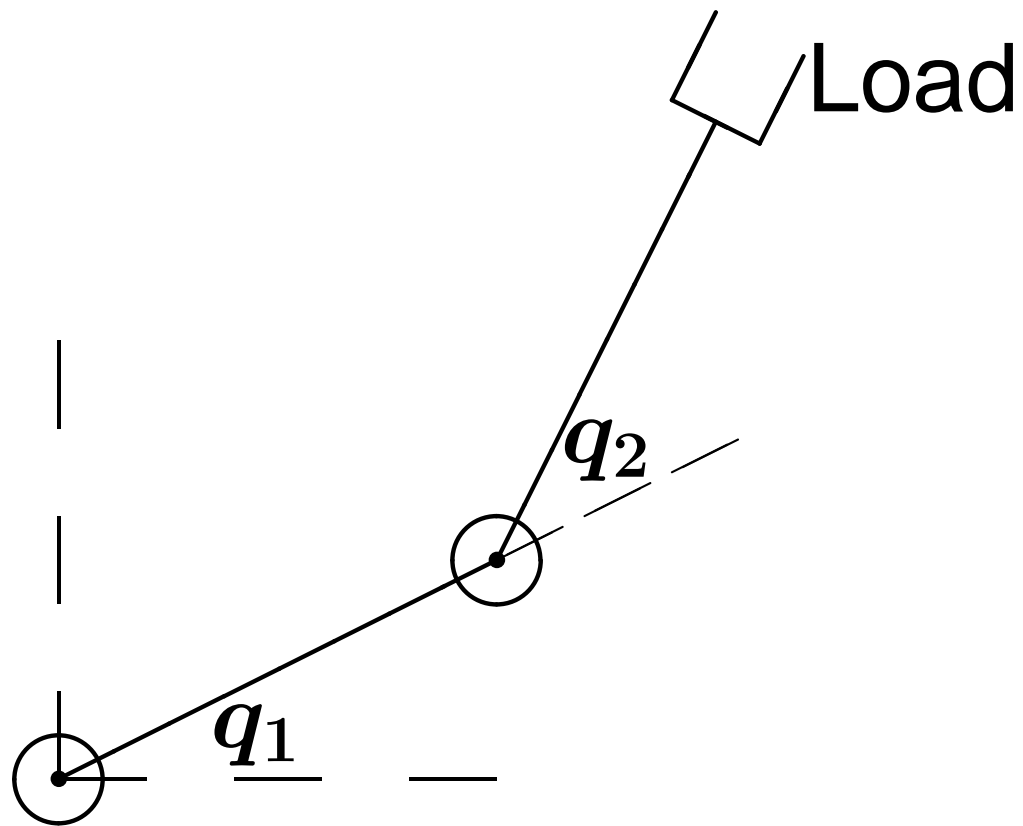The only solution that can stay identically in S is $x(t) \equiv 0$

Thus, the origin is asymptotically stable

Suppose $a = \infty$ and $\int_0^y h_1(z)\, dz \to \infty$ as $|y| \to \infty$

Then, $D = R^2$ and $V(x) = \int_0^{x_1} h_1(y)\, dy + \frac{1}{2}x_2^2$ is radially unbounded. $S = \{x \in R^2 \mid x_2 = 0\}$ and the only solution that can stay identically in S is $x(t) \equiv 0$

The origin is globally asymptotically stable

Example: m-link Robot Manipulator

Load

q_2

q_1

Two-link Robot Manipulator

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D\dot{q} + g(q) = u$$

q is an m-dimensional vector of joint positions

u is an m-dimensional control (torque) inputs

$M = M^T > 0$ is the inertia matrix

$C(q,\dot{q})\dot{q}$ accounts for centrifugal and Coriolis forces

$$(\dot{M} - 2C)^T = -(\dot{M} - 2C)$$

$D\dot{q}$ accounts for viscous damping; $D = D^T \geq 0$

$g(q)$ accounts for gravity forces; $g(q) = [\partial P(q)/\partial q]^T$

$P(q)$ is the total potential energy of the links due to gravity

Investigate the use of the (PD plus gravity compensation) control law

$$u = g(q) - K_p(q - q^*) - K_d\,\dot{q}$$

to stabilize the robot at a desired position q^*, where K_p and K_d are symmetric positive definite matrices

$$e = q - q^*, \quad \dot{e} = \dot{q}$$

$$
\begin{aligned}
M\ddot{e} &= M\ddot{q} \\
&= -C\,\dot{q} - D\,\dot{q} - g(q) + u \\
&= -C\,\dot{q} - D\,\dot{q} - K_p(q - q^*) - K_d\,\dot{q} \\
&= -C\,\dot{e} - D\,\dot{e} - K_p\,e - K_d\,\dot{e}
\end{aligned}
$$

$$M\ddot{e} = -C\,\dot{e} - D\,\dot{e} - K_p\,e - K_d\,\dot{e}$$

$$V = \frac{1}{2}\dot{e}^T M(q)\dot{e} + \frac{1}{2}e^T K_p e$$

$$\dot{V} = \dot{e}^T M\ddot{e} + \frac{1}{2}\dot{e}^T \dot{M}\dot{e} + e^T K_p \dot{e}$$

$$= -\dot{e}^T C\dot{e} - \dot{e}^T D\dot{e} - \dot{e}^T K_p e - \dot{e}^T K_d \dot{e}$$
$$+ \frac{1}{2}\dot{e}^T \dot{M}\dot{e} + e^T K_p \dot{e}$$

$$= \frac{1}{2}\dot{e}^T (\dot{M} - 2C)\dot{e} - \dot{e}^T (K_d + D)\dot{e}$$

$$= -\dot{e}^T (K_d + D)\dot{e} \leq 0$$

$(K_d + D)$ is positive definite

$$\dot{V} = -\dot{e}^T(K_d + D)\dot{e} = 0 \;\Rightarrow\; \dot{e} = 0$$

$$M\ddot{e} = -C\,\dot{e} - D\,\dot{e} - K_p\,e - K_d\,\dot{e}$$

$$\dot{e}(t) \equiv 0 \;\Rightarrow\; \ddot{e}(t) \equiv 0 \;\Rightarrow\; K_p e(t) \equiv 0 \;\Rightarrow\; e(t) \equiv 0$$

By LaSalle's theorem the origin $(e = 0, \dot{e} = 0)$ is globally asymptotically stable