Nonlinear Systems and Control
Lecture # 1
Introduction

Nonlinear State Model

$$
\begin{aligned}
\dot{x}_1 &= f_1(t, x_1, \ldots, x_n, u_1, \ldots, u_p) \\
\dot{x}_2 &= f_2(t, x_1, \ldots, x_n, u_1, \ldots, u_p) \\
\vdots \quad & \quad \vdots \\
\dot{x}_n &= f_n(t, x_1, \ldots, x_n, u_1, \ldots, u_p)
\end{aligned}
$$

$\dot{x}_i$ denotes the derivative of x_i with respect to the time variable t

u_1, u_2, . . ., u_p are input variables

x_1, x_2, . . ., x_n the state variables

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_p \end{bmatrix}, \quad f(t,x,u) = \begin{bmatrix} f_1(t,x,u) \\ f_2(t,x,u) \\ \vdots \\ \vdots \\ f_n(t,x,u) \end{bmatrix}$$

$$\dot{x} = f(t,x,u)$$

$$
\begin{aligned}
\dot{x} &= f(t, x, u) \\
y &= h(t, x, u)
\end{aligned}
$$

x is the state, u is the input
y is the output (q-dimensional vector)

Special Cases:

Linear systems:

$$
\begin{aligned}
\dot{x} &= A(t)x + B(t)u \\
y &= C(t)x + D(t)u
\end{aligned}
$$

Unforced state equation:

$$\dot{x} = f(t, x)$$

Results from $\dot{x} = f(t, x, u)$ with $u = \gamma(t, x)$

Autonomous System:

$$\dot{x} = f(x)$$

Time-Invariant System:

$$\dot{x} = f(x, u)$$
$$y = h(x, u)$$

A time-invariant state model has a time-invariance property with respect to shifting the initial time from t_0 to $t_0 + a$, provided the input waveform is applied from $t_0 + a$ rather than t_0

Existence and Uniqueness of Solutions

$$\dot{x} = f(t, x)$$

$f(t, x)$ is piecewise continuous in t and locally Lipschitz in x over the domain of interest

$f(t, x)$ is piecewise continuous in t on an interval $J \subset R$ if for every bounded subinterval $J_0 \subset J$, f is continuous in t for all $t \in J_0$, except, possibly, at a finite number of points where f may have finite-jump discontinuities

$f(t, x)$ is locally Lipschitz in x at a point x_0 if there is a neighborhood $N(x_0, r) = \{x \in R^n \mid \|x - x_0\| < r\}$ where $f(t, x)$ satisfies the Lipschitz condition

$$\|f(t, x) - f(t, y)\| \ \leq \ L\|x - y\|, \ \ L > 0$$

A function $f(t, x)$ is locally Lipschitz in x on a domain (open and connected set) $D \subset R^n$ if it is locally Lipschitz at every point $x_0 \in D$

When $n = 1$ and f depends only on x

$$\frac{|f(y) - f(x)|}{|y - x|} \leq L$$

On a plot of $f(x)$ versus x, a straight line joining any two points of $f(x)$ cannot have a slope whose absolute value is greater than L

Any function $f(x)$ that has infinite slope at some point is not locally Lipschitz at that point

A discontinuous function is not locally Lipschitz at the points of discontinuity

The function $f(x) = x^{1/3}$ is not locally Lipschitz at $x = 0$ since

$$f'(x) = (1/3)x^{-2/3} \to \infty \text{ as } x \to 0$$

On the other hand, if $f'(x)$ is continuous at a point x_0 then $f(x)$ is locally Lipschitz at the same point because continuity of $f'(x)$ ensures that $|f'(x)|$ is bounded by a constant k in a neighborhood of x_0 ; which implies that $f(x)$ satisfies the Lipschitz condition $L = k$

More generally, if for $t \in J \subset R$ and x in a domain $D \subset R^n$, $f(t, x)$ and its partial derivatives $\partial f_i / \partial x_j$ are continuous, then $f(t, x)$ is locally Lipschitz in x on D

Lemma: Let $f(t, x)$ be piecewise continuous in t and locally Lipschitz in x at x_0, for all $t \in [t_0, t_1]$. Then, there is $\delta > 0$ such that the state equation $\dot{x} = f(t, x)$, with $x(t_0) = x_0$, has a unique solution over $[t_0, t_0 + \delta]$

Without the local Lipschitz condition, we cannot ensure uniqueness of the solution. For example, $\dot{x} = x^{1/3}$ has $x(t) = (2t/3)^{3/2}$ and $x(t) \equiv 0$ as two different solutions when the initial state is $x(0) = 0$

The lemma is a local result because it guarantees existence and uniqueness of the solution over an interval $[t_0, t_0 + \delta]$, but this interval might not include a given interval $[t_0, t_1]$. Indeed the solution may cease to exist after some time

Example:

$$\dot{x} = -x^2$$

$f(x) = -x^2$ is locally Lipschitz for all x

$$x(0) = -1 \;\Rightarrow\; x(t) = \frac{1}{(t-1)}$$

$$x(t) \to -\infty \text{ as } t \to 1$$

the solution has a *finite escape time* at $t = 1$

In general, if $f(t, x)$ is locally Lipschitz over a domain D and the solution of $\dot{x} = f(t, x)$ has a finite escape time t_e, then the solution $x(t)$ must leave every compact (closed and bounded) subset of D as $t \to t_e$

Global Existence and Uniqueness

A function $f(t, x)$ is globally Lipschitz in x if

$$\|f(t, x) - f(t, y)\| \ \leq \ L\|x - y\|$$

for all $x, y \in R^n$ with the same Lipschitz constant L

If $f(t, x)$ and its partial derivatives $\partial f_i/\partial x_j$ are continuous for all $x \in R^n$, then $f(t, x)$ is globally Lipschitz in x if and only if the partial derivatives $\partial f_i/\partial x_j$ are globally bounded, uniformly in t

$f(x) = -x^2$ is locally Lipschitz for all x but not globally Lipschitz because $f'(x) = -2x$ is not globally bounded

Lemma: Let $f(t, x)$ be piecewise continuous in t and globally Lipschitz in x for all $t \in [t_0, t_1]$. Then, the state equation $\dot{x} = f(t, x)$, with $x(t_0) = x_0$, has a unique solution over $[t_0, t_1]$

The global Lipschitz condition is satisfied for linear systems of the form

$$\dot{x} = A(t)x + g(t)$$

but it is a restrictive condition for general nonlinear systems. See Example 3.5.

At the expense of having to know more about the solution of the system, we have the following result.

Lemma: Let $f(t, x)$ be piecewise continuous in t and locally Lipschitz in x for all $t \geq t_0$ and all x in a domain $D \subset R^n$. Let W be a compact subset of D, and suppose that every solution of

$$\dot{x} = f(t, x), \quad x(t_0) = x_0$$

with $x_0 \in W$ lies entirely in W. Then, there is a unique solution that is defined for all $t \geq t_0$

Proof: By the previous lemma (local result), there is a unique local solution over $[t_0, T)$. We want to show $T = \infty$.

Suppose that T is finite, then the solution must leave the compact set W (contradiction). Therefore $T = \infty$.

Check the assumption that every solution lies in a compact set by using Lyapunov method.

Example:

$$\dot{x} = -x^3 = f(x)$$

$f(x)$ is locally Lipschitz on R, but not globally Lipschitz because $f'(x) = -3x^2$ is not globally bounded

If, at any instant of time, $x(t)$ is positive, the derivative $\dot{x}(t)$ will be negative. Similarly, if $x(t)$ is negative, the derivative $\dot{x}(t)$ will be positive

Therefore, starting from any initial condition $x(0) = a$, the solution cannot leave the compact set $\{x \in R \mid |x| \le |a|\}$

Thus, the equation has a unique solution for all $t \ge 0$

Equilibrium Points

A point $x = x^*$ in the state space is said to be an equilibrium point of $\dot{x} = f(t, x)$ if

$$x(t_0) = x^* \;\;\Rightarrow\;\; x(t) \equiv x^*, \;\; \forall\, t \geq t_0$$

For the autonomous system $\dot{x} = f(x)$, the equilibrium points are the real solutions of the equation

$$f(x) = 0$$

An equilibrium point could be isolated; that is, there are no other equilibrium points in its vicinity, or there could be a continuum of equilibrium points

A linear system $\dot{x} = Ax$ can have an isolated equilibrium point at $x = 0$ (if A is nonsingular) or a continuum of equilibrium points in the null space of A (if A is singular)

It cannot have multiple isolated equilibrium points , for if x_a and x_b are two equilibrium points, then by linearity any point on the line $\alpha x_a + (1 - \alpha)x_b$ connecting x_a and x_b will be an equilibrium point

A nonlinear state equation can have multiple isolated equilibrium points . For example, the state equation

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -a \sin x_1 - b x_2$$

has equilibrium points at $(x_1 = n\pi, x_2 = 0)$ for $n = 0, \pm 1, \pm 2, \cdots$

Linearization

A common engineering practice in analyzing a nonlinear system is to linearize it about some nominal operating point and analyze the resulting linear model

What are the limitations of linearization?

- Since linearization is an approximation in the neighborhood of an operating point, it can only predict the "local" behavior of the nonlinear system in the vicinity of that point. It cannot predict the "nonlocal" or "global" behavior
- There are "essentially nonlinear phenomena" that can take place only in the presence of nonlinearity

Nonlinear Phenomena

- Finite escape time: the state of an unstable linear system goes to infinity as time approaches infinity; a nonlinear system's state can go to infinity in finite time
- Multiple isolated equilibrium points: a linear system can have only one isolated equilibrium; a nonlinear system can have multiple isolated equilibrium points
- Limit cycles: a linear system with eigenvalues on the imaginary axis produces an oscillation; a nonlinear system can produce an oscillation of fixed amplitude and frequency
- Subharmonic, harmonic, or almost-periodic oscillations; Chaos; Multiple modes of behavior