Linear Control Systems
Lecture # 8
Observability
&
Discrete-Time Systems
Definition: The system

\[ \dot{x} = Ax, \quad y = Cx \]

or the pair \((A, C)\), is said to be observable on \([t_0, t_f]\) if any initial state \(x(t_0) = x_0\) can be uniquely determined from \(y(t)\) on \([t_0, t_f]\). It is said to be (re)constructible on \([t_0, t_f]\) if any final state \(x(t_f) = x_f\) can be uniquely determined from \(y(t)\) on \([t_0, t_f]\).

Without loss of generality, take \(t_0 = 0\)

The observability Gramian of \((A, C)\) is defined by

\[ W_o(0, t_f) = \int_{0}^{t_f} e^{A^T t} C^T C e^{A t} \, dt \]
Lemma: $W_o(0, t_f)$ is positive definite if and only if there is no vector $x_a \neq 0$ such that

$$Ce^{At}x_a \equiv 0, \quad \forall \ t \in [0, t_f]$$

Proof:

$$x_a^T W_o(0, t_f) x_a = \int_0^{t_f} x_a^T e^{A^Tt} C^T Ce^{At} x_a \ dt$$

$$x_a^T W_o(0, t_f) x_a = 0 \iff Ce^{At}x_a \equiv 0, \quad \forall \ t \in [0, t_f]$$
**Theorem:** The pair \((A, C)\) is observable (constructible) on \([0, t_f]\) if and only if the observability Gramian \(W_o(0, t_f)\) is positive definite

**Proof of sufficiency:** Suppose \(W_o(0, t_f)\) is positive definite and let \(x_0\) be the initial state

\[
y(t) = Cx(t) = Ce^{At}x_0
\]

Multiply from the left by \(e^{A^Tt}C^T\)

\[
e^{A^Tt}C^Ty(t) = e^{A^Tt}C^Ce^{At}x_0
\]

Integrate from 0 to \(t_f\)

\[
\int_0^{t_f} e^{A^Tt}C^Ty(t) \, dt = \int_0^{t_f} e^{A^Tt}C^Ce^{At} \, dt \, x_0
\]
\[ \int_{0}^{t_f} e^{A^T t} C^T y(t) \, dt = W_0(0, t_f) x_0 \]

This equation has a unique solution

\[ x_0 = W_0^{-1}(0, t_f) \int_{0}^{t_f} e^{A^T t} C^T y(t) \, dt \]

Since

\[ x(t_f) = e^{A t_f} x_0 \]

and \( e^{A t_f} \) is nonsingular, \( x(t_f) \) is determined uniquely by

\[ x(t_f) = e^{A t_f} W_0^{-1}(0, t_f) \int_{0}^{t_f} e^{A^T t} C^T y(t) \, dt \]
Proof of Necessity: We want to show that positive definiteness of $W_o(0, t_f)$ is a necessary condition for observability (constructibility) over $[0, t_f]$. Suppose $W_o(0, t_f)$ is not positive definite. Then, there is a vector $x_a \neq 0$ such that

$$Ce^{At}x_a \equiv 0, \quad \forall \ t \in [0, t_f]$$

For the case of observability, the output due to $x(0) = x_0$ is

$$Ce^{At}x_0$$

and the output due to $x(0) = x_0 + x_a$ is

$$Ce^{At}(x_0 + x_a) = Ce^{At}x_0 + Ce^{At}x_a = Ce^{At}x_0$$
The initial states $x_0$ and $x_0 + x_a$ produce the same output. Hence, $x_0$ cannot be uniquely determined from the output.

For the case of constructibility,

$$y(t) = Ce^{At}x(0) = Ce^{At}e^{-At_f}x(t_f)$$

$x(t_f) = x_f$ and $x(t_f) = x_f + e^{At_f}x_a$ correspond to the same output because

$$Ce^{At}e^{-At_f} \left( x_f + e^{At_f}x_a \right) = Ce^{At}e^{-At_f}x_f$$

$$+ Ce^{At}e^{-At_f}e^{At_f}x_a$$

$$= Ce^{At}e^{-At_f}x_f + Ce^{At}x_a$$

$$= Ce^{At}e^{-At_f}x_f$$
Lemma: The observability Gramian $W_o(0, t_f)$ is positive definite if and only if $\text{rank } \mathcal{O} = n$, where

\[
\mathcal{O} = \begin{bmatrix}
C \\
CA \\
\vdots \\
CA^{n-1}
\end{bmatrix}
\]

is the observability matrix ($\mathcal{O}$ is $np \times n$ when $A$ is $n \times n$ and $C$ is $p \times n$)
Proof:

$W_0(0, t_f)$ is singular if and only if

there is $x_a \neq 0$ such that $C e^{At} x_a \equiv 0, \ \forall \ t \in [0, t_f]$

$\Leftrightarrow C \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k x_a \equiv 0, \ \forall \ t \in [0, t_f]$

$\Leftrightarrow C A^k x_a = 0, \ \forall \ k \geq 0$

$\Leftrightarrow C A^k x_a = 0, \ \forall \ k = 0, 1, \ldots, n - 1$
Theorem: The pair \((A, C)\) is observable (reconstructible) if and only if \(\text{rank } \mathcal{O} = n\)
Duality

\((A, B)\) controllable \iff \text{rank} \begin{bmatrix} B, AB, \ldots, A^{n-1}B \end{bmatrix} = n

\iff \text{rank} \begin{bmatrix} B^T \\ B^T A^T \\ \vdots \\ B^T (A^T)^{n-1} \end{bmatrix} = n

\iff (A^T, B^T) \text{ observable}

Similarly, observability of \((A, C)\) is equivalent to controllability of \((A^T, C^T)\)
Controllability and observability are invariant to state transformations

\[ \{A, B, C, D\} \rightarrow \boxed{x = Pz} \rightarrow \{P^{-1}AP, P^{-1}B, CP, D\} \]

\[
\begin{align*}
\text{rank} & \left[ P^{-1}B, P^{-1}APP^{-1}B, \ldots, (P^{-1}AP)^{n-1} P^{-1}B \right] \\
& = \text{rank} \left[ P^{-1}B, P^{-1}AB, \ldots, P^{-1}A^{n-1}B \right] \\
& = \text{rank} \left\{ P^{-1}[B, AB, \ldots, A^{n-1}B] \right\} \\
& = \text{rank} \left[ B, AB, \ldots, A^{n-1}B \right]
\end{align*}
\]

Matlab:

\[ \mathcal{O} = \text{obsv}(A, C); \quad \text{rank}(\mathcal{O}) \]
Mathematical Preliminaries: Consider the equation

\[ y = Mx \]

where \( M \) is an \( m \times n \) matrix. The equation has a solution if and only if \( y \) can be expressed as a linear combination of the columns of \( M \)

\[ \text{rank } M = \text{rank } [M, y] \]

The range space of \( M \), denoted by \( \mathcal{R}(M) \), is the set of all linear combinations of the columns of \( M \). It is a subspace of \( \mathbb{R}^m \) of dimension equal to \( \text{rank } M \). The equation \( y = Mx \) has a solution if and only if \( y \in \mathcal{R}(M) \). The equation \( y = Mx \) has a solution for every \( y \in \mathbb{R}^m \) if and only if

\[ \text{rank}(M) = m \iff \mathcal{R}(M) = \mathbb{R}^m \]
Given that \( y = Mx \) has a solution. Is the solution unique? Suppose there are two solutions \( x_a \) and \( x_b \) such that

\[
y = Mx_a, \quad y = Mx_b
\]

\[
M(x_a - x_b) = 0
\]

The null space of \( M \), denoted by \( \mathcal{N}(M) \), is the set of all vectors \( x \in \mathbb{R}^n \) such that \( Mx = 0 \). It is a subspace of \( \mathbb{R}^n \) of dimension \( (n - \text{rank } M) \)

The solution of \( y = Mx \) is unique if and only if there is no vector \( x \neq 0 \) such that \( Mx = 0 \)

Equivalently, \( \text{rank } M = n \)
The definitions of controllability (reachability) and observability (constructibility) are the same as in the continuous-time case.

Let us study steering the state of the system

\[ x(k + 1) = Ax(k) + Bu(k) \]

Find \( u(k) \) to steer the state of the system from \( x_0 \) to \( x_f \) in finite time

\[ x(k) = A^k x(0) + \sum_{j=0}^{k-1} A^{k-j-1} Bu(j) \]
\[ x(k) - A^k x_0 = [B, AB, \ldots, A^{k-1}B] \begin{bmatrix} u(k - 1) \\ u(k - 2) \\ \vdots \\ u(0) \end{bmatrix} \]

Can we choose \( u(0), \ldots, u(k - 1) \) such that \( x(k) = x_f \) for some \( k \)?

The answer is Yes if for some \( k \),

\[ x_f - A^k x_0 \in \mathcal{R} \left( [B, AB, \ldots, A^{k-1}B] \right) \]
Due to Cayley-Hamilton theorem

\[ \mathcal{R} \left( [B, AB, \ldots, A^{k-1}B] \right) = \mathcal{R} \left( [B, AB, \ldots, A^{n-1}B] \right) \]

for all \( k \geq n \)

Thus, we take \( k = n \) and rewrite the equation as

\[ x_f - A^{n-1}x_0 = C \begin{bmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(0) \end{bmatrix} \overset{\text{def}}{=} C U \]
Reachability: \( x_0 = 0 \) and \( x_f \) is any vector in \( \mathbb{R}^n \)

\[ x_f = C \ U \]

There is a solution for every \( x_f \) if and only if

\[ \mathcal{R}(C) = \mathbb{R}^n \iff \text{rank} \ C = n \]

Controllability: \( x_f = 0 \) and \( x_0 \) is any vector in \( \mathbb{R}^n \)

\[ -A^n x_0 = C \ U \]

There is a solution if and only if \( -A^n x_0 \in \mathcal{R}(C) \)

\( \text{rank} \ C = n \) is a sufficient condition for the existence of a solution. Is it necessary?
If $A$ is nonsingular, then the vector $A^n x_0$ can be any vector in $\mathbb{R}^n$. In this case, the condition \( \text{rank } C = n \) is necessary.

If $A$ is singular, the equation

$$-A^n x_0 = C U$$

may be solvable even if \( \text{rank } C < n \).
Example:

\[ A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

\[ C = [B, AB] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \]

\[ \text{rank } C = 1 \]
\[
\begin{bmatrix}
\alpha + \beta \\
0
\end{bmatrix} = \begin{bmatrix} 1 & 1 \\
0 & 0 \end{bmatrix} \begin{bmatrix} u(1) \\
u(0) \end{bmatrix}
\]

\[-(\alpha + \beta) = u(0) + u(1)\]

There are infinitely many solutions. Two examples are

\[u(0) = -\alpha - \beta, \quad u(1) = 0\]

\[u(0) = -\alpha, \quad u(1) = -\beta\]
Summary:

- \( \text{rank } C = n \) is a necessary and sufficient condition for reachability of \( x(k+1) = Ax(k) + Bu(k) \)
- \( \text{rank } C = n \) is a sufficient condition for controllability of \( x(k+1) = Ax(k) + Bu(k) \)
- If \( A \) is nonsingular, then \( \text{rank } C = n \) is a necessary and sufficient condition for controllability of \( x(k+1) = Ax(k) + Bu(k) \)

Remark: Strictly speaking, we should say that the pair \((A, B)\) is reachable if and only \( \text{rank } C = n \). However, by abuse of notation, we will continue to call the pair \((A, B)\) controllable if and only if \( \text{rank } C = n \). This will keep the discussion of controllability the same for both continuous-time and discrete-time systems.
Observability: Consider the system

\[ x(k + 1) = Ax(k), \quad y(k) = Cx(k) \]

and study the question of uniquely determining \( x(0) \) from the output \( y(k) \)

\[
\begin{bmatrix}
  y(0) \\
  y(1) \\
  \vdots \\
  y(k - 1)
\end{bmatrix} =
\begin{bmatrix}
  Cx(0) \\
  CAx(0) \\
  \vdots \\
  CA^{k-1}x(0)
\end{bmatrix} =
\begin{bmatrix}
  C \\
  CA \\
  \vdots \\
  CA^{k-1}
\end{bmatrix}x(0)
\]
The equation will have a unique solution $x(0)$ if and only if

$$\text{rank} \begin{bmatrix}
    C \\
    CA \\
    \vdots \\
    CA^{k-1}
\end{bmatrix} = n$$

By Cayley-Hamilton theorem

$$\text{rank} \begin{bmatrix}
    C \\
    CA \\
    \vdots \\
    CA^{k-1}
\end{bmatrix} = \text{rank} \begin{bmatrix}
    C \\
    CA \\
    \vdots \\
    CA^{n-1}
\end{bmatrix}, \quad \forall \; k \geq n$$
Take $k = n$ and write the equation as

$$Y = \mathcal{O} \ x(0)$$

where

$$Y = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(n-1) \end{bmatrix}$$

The equation has a unique solution if and only if $\text{rank } \mathcal{O} = n$