Linear Control Systems
Lecture # 5
Stability and Matlab
Internal Stability

Internal stability deals with the boundedness and asymptotic behavior (as $t \to \infty$) of the solutions of

$\dot{x} = Ax$

The solution of $\dot{x} = Ax$ is given by

$x(t) = e^{At}x(0)$

**Definition:** The system $\dot{x} = Ax$ is stable if

$\|e^{At}\| \leq \gamma, \quad \forall \ t \geq 0, \ \gamma > 0$

and asymptotically (or exponentially) stable if

$\|e^{At}\| \leq \gamma e^{-\lambda t}, \quad \forall \ t \geq 0, \ \gamma > 0, \ \lambda > 0$
If $A$ is diagonalizable

$$e^{At} = \sum_{i=1}^{n} e^{\lambda_i t} v_i w_i^T$$

The behavior depends on $Re[\lambda_i]$

$$Re[\lambda_i] < 0 \Rightarrow |e^{\lambda_i t}| \leq e^{-\lambda t}, \lambda > 0$$

$$Re[\lambda_i] = 0 \Rightarrow |e^{\lambda_i t}| = 1$$

$$Re[\lambda_i] > 0 \Rightarrow e^{\lambda_i t} \text{ is unbounded}$$
In general

\[ e^{At} = \sum_{i=1}^{r} \sum_{k=1}^{m_i} W_{ik} \frac{t^{k-1}}{(k-1)!} e^{\lambda_i t} \]

\[ R_e[\lambda_i] > 0 \Rightarrow t^{k-1} e^{\lambda_i t} \text{ is unbounded} \]

\[ R_e[\lambda_i] < 0 \Rightarrow |t^{k-1} e^{\lambda_i t}| \leq \gamma e^{-\lambda t}, \gamma > 0, \lambda > 0 \]

\[ R_e[\lambda_i] = 0 \text{ and } k = 1 \Rightarrow e^{\lambda_i t} \text{ is bounded} \]

\[ R_e[\lambda_i] = 0 \text{ and } k \geq 2 \Rightarrow t^{k-1} e^{\lambda_i t} \text{ is unbounded} \]
When will the dimension of the Jordan block $J_i$ be higher than one?

$$(A - \lambda_i I)v_i = 0$$

Let $q_i$ be the algebraic multiplicity of $\lambda_i$. If

$$q_i = \text{nullity}(A - \lambda_i I) = n - \text{rank}(A - \lambda_i I)$$

then there are $q_i$ linearly independent eigenvectors associated with the eigenvalue $\lambda_i$. If this is true for every eigenvalue that has multiplicity higher than one, then we can find $n$ linearly independent eigenvectors and $A$ is diagonalizable.
If
\[
\text{nullity}(A - \lambda_i I) = n - \text{rank}(A - \lambda_i I) < q_i
\]
for any eigenvalue with multiplicity higher than one, then $A$ is not diagonalizable.

nullity($A - \lambda_i I$) is equivalent to the geometric multiplicity of the eigenvalue $\lambda_i$.

We need
\[
q_i - \text{nullity}(A - \lambda_i I)
\]
number of generalized eigenvectors
Theorem: The system $\dot{x} = Ax$ is

- Stable if and only if

$$Re[\lambda_i] \leq 0, \text{ for } i = 1, 2, \ldots, n$$

and for every eigenvalue with $Re[\lambda_i] = 0$ and algebraic multiplicity $q_i \geq 2$,

$$\text{rank}(A - \lambda_i I) = n - q_i$$

- Asymptotically (or exponentially) stable if

$$Re[\lambda_i] < 0, \text{ for } i = 1, 2, \ldots, n$$
Example: Study the stability of $\dot{x} = Ax$, where

$$A = \begin{bmatrix} -1 & 0 & 3 & 1 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

Eigenvalues: $-1, -2, -3, -3$

The system is asymptotically stable
Example: Study the stability of $\dot{x} = Ax$, where

$$A = \begin{bmatrix}
-1 & 0 & 3 & 1 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}$$

Eigenvalues: $-1, 1, 0, 0$

The system is unstable
Example: Consider the series and parallel connections of two identical systems, each represented by

\[
\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x
\]

or

\[
H(s) = \frac{1}{s^2 + 1}
\]

\[
A_s = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}, \quad A_p = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}
\]

Eigenvalues: \( \pm j, \pm j \)
Series Case:

\[ \lambda_1 = j, \; q_1 = 2, \; n - q_1 = 2 \]

\[ A_s - \lambda_1 I = \begin{bmatrix} -j & 1 & 0 & 0 \\ -1 & -j & 0 & 0 \\ 0 & 0 & -j & 1 \\ 1 & 0 & -1 & -j \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -j & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -j \end{bmatrix} \]

\[ \text{rank}(A_s - \lambda_1 I) = 3 \]

The system is unstable
Parallel Case:

\[ \lambda_1 = j, \quad q_1 = 2, \quad n - q_1 = 2 \]

\[ A_p - \lambda_1 I = \begin{bmatrix} -j & 1 & 0 & 0 \\ -1 & -j & 0 & 0 \\ 0 & 0 & -j & 1 \\ 0 & 0 & -1 & -j \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -j & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -j \end{bmatrix} \]

\[ \text{rank}(A_p - \lambda_1 I) = 2 \]

The system is stable
What is the effect of state transformations on stability?

\[ A \rightarrow x = Pz \rightarrow P^{-1}AP \]

\[ Av = \lambda v \]

\[ P^{-1}Av = \lambda P^{-1}v \]

\[ (P^{-1}AP)(P^{-1}v) = \lambda P^{-1}v \]

\( A \) and \( P^{-1}AP \) have the same eigenvalues. If \( v_i \) is an eigenvector of \( A \), then \( P^{-1}v_i \) is an eigenvector of \( P^{-1}AP \)

\[ \text{rank} \left( P^{-1}AP - \lambda I \right) = \text{rank} \left[ P^{-1}(A - \lambda I)P \right] = \text{rank} \left( A - \lambda I \right) \]

Internal stability is invariant under state transformations
Input-Output Stability

Definition: The linear system $\hat{y}(s) = H(s)\hat{u}(s)$ is Bounded-Input Bounded-Output (BIBO) stable if for every bounded input $u(t)$, the output $y(t)$ is bounded. Equivalently, for every $k_u > 0$ there is $k_y > 0$ such that

$$\|u(t)\| \leq k_u, \quad \forall \ t \geq 0 \ \Rightarrow \ \|y(t)\| \leq k_y, \quad \forall \ t \geq 0$$

By taking the inverse Laplace transform

$$y(t) = \int_0^t H(t - \tau)u(\tau) \, d\tau$$

where $H(t) = \mathcal{L}^{-1}\{H(s)\}$ is the impulse response matrix.
Theorem: The system $\hat{y}(s) = H(s)\hat{u}(s)$ is BIBO stable if and only if
\[ \int_{0}^{\infty} \|H(t)\| \, dt < \infty \]

Proof of sufficiency:
\[
\|y(t)\| = \left\| \int_{0}^{t} H(t - \tau)u(\tau) \, d\tau \right\|
\leq \int_{0}^{t} \|H(t - \tau)\| \, d\tau \|u(\tau)\|
\leq \int_{0}^{\infty} \|H(t - \tau)\| \, d\tau \, k_u
\leq \int_{0}^{\infty} \|H(\sigma)\| \, d\sigma \, k_u \overset{\text{def}}{=} k_y
\]
Proof on Necessity: Use a contradiction argument (in the SISO case). Given $k_u > 0$, suppose there is $k_y > 0$ such that

$$|u(t)| \leq k_u, \quad \forall \ t \geq 0 \Rightarrow |y(t)| \leq k_y, \quad \forall \ t \geq 0$$

but $\int_0^\infty |h(t)| \ dt$ is not finite

There is $t_1$ (dependent on $k_y/k_u$) such that

$$\int_0^{t_1} |h(t_1 - \tau)| \ d\tau > \frac{k_y}{k_u}$$
Let

\[ u(t) = \begin{cases} 
  k_u, & \text{when } h(t_1 - t) > 0 \\
  0, & \text{when } h(t_1 - t) = 0 \\
  -k_u, & \text{when } h(t_1 - t) < 0 
\end{cases} \]

\[ |u(t)| \leq k_u, \quad \text{for } 0 \leq t \leq t_1 \]

\[ y(t_1) = \int_0^{t_1} h(t_1 - \tau) u(\tau) \, d\tau = \int_0^{t_1} k_u |h(t_1 - \tau)| \, d\tau > k_y \]

Contradiction
Example: Time delay element

\[ y(t) = u(t - T) \]

\[ H(s) = e^{-sT} \]

\[ h(t) = \mathcal{L}^{-1}\{H(s)\} = \delta(t - T) \]

\[ |u(t)| \leq k_u \Rightarrow |y(t)| \leq k_u \]

Or

\[ \int_{0}^{\infty} \delta(t - T) \, dt = 1 \]
When

\[ H(s) = C(sI - A)^{-1}B + D \]

the elements \( h_{ij}(s) \) of \( H(s) \) are proper rational functions of \( s \)

\[ h_{ij}(s) = \frac{n_{ij}(s)}{d_{ij}(s)} \]

where \( n_{ij}(s) \) are \( d_{ij}(s) \) are polynomials with

\[ \text{deg}(n_{ij}) \leq \text{deg}(d_{ij}) \]
Since

\[(sI - A)^{-1} = \frac{1}{\det(sI - A)} \text{Adjoint}(sI - A)\]

the poles of \(h_{ij}(s)\) are roots of

\[\det(sI - A) = 0\]

that is, eigenvalues of \(A\)

Not all eigenvalues of \(A\) will appear as poles of some elements of \(H(s)\) because some eigenvalues could be cancelled
Given a strictly proper rational function $H(s)$, let $h(t) = \mathcal{L}^{-1}\{H(s)\}$. When will

$$\int_0^\infty |h(t)| \, dt < \infty$$

$H(s)$ can be expressed as the sum of terms of the form

$$\frac{K}{(s - p)^\alpha}$$

$$\mathcal{L}^{-1}\left\{\frac{K}{(s - p)^\alpha}\right\} = K \frac{t^{\alpha-1}}{\alpha - 1)!} e^{pt}$$

**Theorem:** $H(s)$ is BIBO stable if and only if all poles of every element of $H(s)$ have negative real parts
What is the relationship between asymptotic and BIBO stability?

The system $\dot{x} = Ax$ is asymptotically stable if all the eigenvalues of $A$ have negative real parts.

The system $H(s) = C(sI - A)^{-1}B + D$ is BIBO stable if all poles of all elements of $H(s)$ have negative real parts.

The poles of $H(s)$ are eigenvalues of $A$.

Asymptotic stability $\Rightarrow$ BIBO stability

What about the opposite implication?

Some eigenvalues of $A$ may not appear as poles of $H(s)$. If such eigenvalues have nonnegative real parts, then we could have a situation where the system is BIBO stable but not asymptotically stable.
Example: Suppose $A$ is diagonalizable

$$e^{At} = \sum_{i=1}^{n} e^{\lambda_i t} v_i w_i^T$$

$$(sI - A)^{-1} = \mathcal{L} \{ e^{At} \} = \sum_{i=1}^{n} \frac{1}{s - \lambda_i} v_i w_i^T$$

$$H(s) = C(sI - A)^{-1} B + D = \sum_{i=1}^{n} \frac{1}{s - \lambda_i} Cv_i w_i^T B + D$$

If $Cv_i = 0$ or $w_i^T B = 0$, the eigenvalue $\lambda_i$ cancels out of $H(s)$
\[
\{A, B, C, D\} \rightarrow x = Pz \rightarrow \{\Lambda, P^{-1}B, CP, D\}
\]
\[
\dot{x} = Ax + Bu \rightarrow \dot{z} = \Lambda z + (P^{-1}B := \tilde{B})u
\]
\[
y = Cx + Du \rightarrow y = (CP := \tilde{C})z + Du
\]
\[
\Lambda = \begin{bmatrix}
\lambda_1 \\
\vdots \\
\lambda_n
\end{bmatrix}, P = [v_1, \ldots, v_n], P^{-1} =: \begin{bmatrix}
w_1^T \\
\vdots \\
w_n^T
\end{bmatrix}
\]
\[
CP = [Cv_1, \ldots, cv_n] =: [\tilde{c}_1, \ldots, \tilde{c}_n]
\]
\[
P^{-1}B = \begin{bmatrix}
w_1^TB \\
\vdots \\
w_n^TB
\end{bmatrix} =: \begin{bmatrix}
\tilde{b}_1 \\
\vdots \\
\tilde{b}_n
\end{bmatrix}
\]
Uncontrollable Mode and Unobservable Mode

\[ H(s) = C(sI - A)^{-1}B = \sum_{i=1}^{n} \frac{1}{s - \lambda_i} \tilde{c}_i \tilde{b}_i \]
Example: Suppose we want to stabilize an unstable system described by

\[ G_p(s) = \frac{1}{s - 1} \]

Consider a cascade compensator

\[ G_c(s) = \frac{s - 1}{s + 1} \]

so that

\[ G_p(s)G_c(s) = \frac{1}{s - 1} \cdot \frac{s - 1}{s + 1} = \frac{1}{s + 1} \]

The system is BIBO stable. Is it asymptotically stable?
Find a state model of the system

\[ G_p(s) = \frac{1}{s - 1} \quad \Rightarrow \quad \dot{x}_1 = x_1 + v, \quad y = x_1 \]

\[ G_c(s) = \frac{s - 1}{s + 1} \quad \Rightarrow \quad \dot{x}_2 = -x_2 + u, \quad v = -2x_2 + u \]

\[ \dot{x}_1 = x_1 - 2x_2 + u \]

\[ \dot{x} = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \]
The system is not asymptotically stable. The eigenvalue +1 is called a hidden mode. Unstable hidden modes are not acceptable because they can be excited by initial conditions or disturbances. For example

$$x_1(0) = \alpha, \ x_2(0) = 0, \ u(t) \equiv 0 \Rightarrow x_2(t) \equiv 0$$

$$\Rightarrow x_1(t) = \alpha e^t$$