ME 812
Conductive Heat Transfer

Separation of Variable Method to Solve Partial Differential Equations.

1. Requirements for the Differential Equation and boundary Conditions.
   a. Differential Equation must be homogeneous
   b. All but one of the boundary conditions must be homogeneous
   c. Variable coefficient’s must be separable. That is, they must be able to be expressed as products functions of the independent variables singularly.
   Example: An appropriate form would be
   \[
   \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0
   \]
   with boundary conditions
   \[
   T(x=0,y) = 0 \\
   T(x=a,y) = T_3 \\
   T(x,y=0) = 0 \\
   T(x,y=b) = 0
   \]
   Equations that do not satisfy this form may be transformed to a number of separate problems of the appropriate form by using superposition.

2. Assume a solution of the form
   \[
   T = \Gamma(x)\cdot\Lambda(y)
   \]

3. Substitute into the differential equation
   \[
   \Lambda \frac{d^2 \Gamma}{dx^2} + \Gamma \frac{d^2 \Lambda}{dy^2} = 0
   \]

4. Substitute into the boundary conditions
   \[
   T(x=0,y) = \Lambda(y)\Gamma(0) = 0 \Rightarrow \Gamma(0) = 0 \\
   T(x=a,y) = \Lambda(y)\Gamma(a) = T_3
   \]
\[ T(x, y=0) = \Lambda(0) \Gamma(x) = 0 \Rightarrow \Lambda(0) = 0 \]
\[ T(x, y=b) = \Lambda(b) \Gamma(x) = 0 \Rightarrow \Lambda(b) = 0 \]

5. Rearrange the differential equation

\[ \frac{1}{\Gamma} \frac{d^2 \Gamma}{dx^2} + \frac{1}{\Lambda} \frac{d^2 \Lambda}{dy^2} = 0 \]

6. Since both terms are only a function of their respective independent variable, then only way to satisfy the equation is for one to be equal to a positive constant and one equal to the negative of that constant, or

\[ \frac{1}{\Gamma} \frac{d^2 \Gamma}{dx^2} = \pm \mu^2 \]
\[ \frac{1}{\Lambda} \frac{d^2 \Lambda}{dy^2} = \mp \mu^2 \]

7. To chose the sign for each term, we rewrite them into ode form

\[ \frac{d^2 \Gamma}{dx^2} - \pm \mu^2 \Gamma = 0 \]
\[ \frac{d^2 \Lambda}{dy^2} - \mp \mu^2 \Lambda = 0 \]

If we have a minus sign between the 2\textsuperscript{nd} derivative term and the dependent variable term then our solution would \pm exponentials or cosh and sinh. If the sign is positive we would get cosines and sines. We choose the sign so that the direction which has the two homogenous boundary conditions gets the sines and cosines, since we must have periodic like functions in that direction to satisfy two homogenous boundary conditions. In general we would say that the directions with two homogenous boundary condition must yield periodic, orthogonal functions, like cosines and sines. Then for our example we will have
\[
\frac{d^2 \Gamma}{dx^2} - \mu^2 \Gamma = 0
\]
\[
\frac{d^2 \Lambda}{dy^2} + \mu^2 \Lambda = 0
\]

8. Solve the ODE’s.

\[
\Gamma(x) = A \cosh(\mu x) + B \sinh(\mu x)
\]
\[
\Lambda(y) = C \cos(\mu y) + D \sin(\mu y)
\]

9. Apply the homogeneous boundary conditions.

\[
\Gamma(x = 0) = 0 = A \cosh(0) + B \sinh(0) \implies A = 0
\]
\[
\Lambda(y = 0) = 0 = C \cos(0) + D \sin(0) \implies C = 0
\]
\[
\Lambda(y = b) = 0 = D \sin(\mu b)
\]

We note that with our boundary condition at \( y = b \), we cannot let \( D = 0 \) or we end up with the only admissible solution being the trivial one, \( T = 0 \), which is not acceptable. However, recall that our sine function is periodic, so we can chose \( \mu \) so as to satisfy the boundary condition. We also note that there is not just one \( \mu \), but several that could fit the bill. Then

\[
\mu = \frac{n\pi}{b}
\]

These are often called the eigen values of the problem and the corresponding functions the eigen functions.

10. Reassemble our solution, recognizing that due to the periodicity condition we now must a linear combination of several functions for our solution or

\[
T(x, y) = \Gamma(x)\Lambda(y) = \sum_{n=1}^{\infty} E_n \sinh(\mu_n x)\sin(\mu_n y)
\]

where \( \mu_n = \frac{n\pi}{b} \)
11. Apply the non-homogeneous boundary condition.

\[ T(x = a, y) = \sum_{n=1}^{\infty} E_n \sinh(\mu_n a) \sin(\mu_n y) = T_3 \]

We need to use this boundary condition to specify our constants \( E_n \). However, to do this we must generate a separate equation for each one. We can do this by recognizing that sine has a special property called orthogonality where

\[
\int_0^{\pi} \sin(nz) \sin(mz) \, dz = \begin{cases} 
\pi/2 & \text{for } m = n \\
0 & \text{for } m \neq n 
\end{cases}
\]

Hence by applying orthogonality we will be able to pluck out each \( E \) one at a time. To do this we must transform the property into a form that will be useful for the problem at hand. Comparing the orthogonality condition with our problem we see that

\[ z = \frac{\pi x}{b} \]

So that our orthogonality condition becomes

\[
\int_0^{b} \sin\left(\frac{n\pi x}{b}\right) \sin\left(\frac{m\pi x}{b}\right) \, dx = \begin{cases} 
b/2 & \text{for } m = n \\
0 & \text{for } m \neq n 
\end{cases}
\]

To obtain the \( m \)th constant, \( E_m \), in our series, we take our non-homogenous boundary condition, multiply by \( \sin(m\pi x/b) \) and integrate from 0 to \( b \),

\[
\int_{0}^{b} \sum_{n=1}^{\infty} E_n \sinh(\mu_n a) \sin(\mu_n y) \sin(\mu_m y) \, dy = \int_{0}^{b} T_3 \sin(\mu_m y) \, dy
\]

We note that only the \( n=m \) is non zero, so that

\[ E_m \sinh(\mu_m a) \frac{b}{2} = \frac{T_3}{\mu_m} \left[ \cos(\mu_m b) - 1 \right] \]
where
\[
\cos(\mu_m b) = \cos\left(\frac{m\pi}{b}\right) = (-1)^m
\]

Then
\[
E_m = \frac{2T_3}{b\mu_m \sinh(\mu_m a)} \left[ 1 - (-1)^m \right]
\]

12. Reassemble final solution
\[
T(x, y) = \sum_{n=1}^{\infty} \frac{2T_3}{b\mu_n} \left[ 1 - (-1)^m \right] \frac{\sinh(\mu_n x)}{\sinh(\mu_n a)} \sin(\mu_n y)
\]

with \(\mu_n = \frac{n\pi}{b}\)