A differential equation of the form

$$\frac{d}{dx} \left[ (1 - x^2) \frac{du}{dx} \right] + n(n + 1)u = 0$$

can be solved by the power series approach. That is, we assume a solution of the form

$$u(x) = \sum_{n=0}^{\infty} a_n x^n$$

substitute and obtain expressions for the $a_n$'s. For the above differential equation the resulting power series are called the Legendre polynomials. The two linearly independent solutions to the equation are the Legendre polynomial of degree $n$ of the first kind, $P_n(x)$, and the Legendre polynomial of degree $n$ of the second kind, $Q_n(x)$. They are given by Rodrigues' formula:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left( x^2 - 1 \right)^n$$

For $n=0$, we would have

$$P_0(x) = \frac{1}{2^0 0!} \frac{d^0}{dx^0} \left( x^2 - 1 \right)^0 = 1$$

For $n=1$, we have

$$P_1(x) = \frac{1}{2^1 1!} \frac{d}{dx} \left( x^2 - 1 \right) = \frac{2x}{2} = x$$

There is a very useful recursive formula of the form

$$P_{n+1}(x) = \frac{(2n + 1)xP_n(x) - nP_{n-1}(x)}{(n + 1)}$$

For $n=1$, this gives

$$P_2(x) = \frac{3xP_1(x) - P_0(x)}{2} = \frac{3x^2 - 1}{2} = \frac{3}{2}x^2 - 1/2$$
Similarly we can develop

\[
P_3(x) = \frac{5}{2}x^3 - 3/2x
\]

\[
P_4(x) = \frac{35}{8}x^4 - 30/8x^2 + 3/8
\]

\[
P_5(x) = \frac{1}{8} \left( 63x^5 - 70x^3 + 15x \right)
\]

A graph of these polynomials is given below:

We note that the following observations:
- P’s are bounded between -1 and 1
- All P’s are 1 at x=1
- Odd P’s are zero at x=0
- Odd P’s are -1 at x=-1
- Even P’s are 1 at x=-1
- Number of zeros (times P crosses x-axis) is equal to n.

For the Legendre polynomials of the second kind (the Q’s), we have
\[ Q_0(x) = \begin{cases} 
\frac{1}{2} \ln \frac{1+x}{1-x} & \text{for } |x| < 1 \\
\frac{1}{2} \ln \frac{x+1}{x-1} & \text{for } |x| > 1
\end{cases} \]

with a recursive formula

\[ Q_n(x) = Q_0(x)P_n(x) - \frac{(2n-1)}{n} P_{n-1}(x) - \frac{(2n-5)}{3(n-1)} P_{n-2}(x) - \ldots \]

Below is a graph of the Legendre polynomials of the 2\textsuperscript{nd} kind.

We note that the following observations:

- All Q’s go to \( \infty \) at x=1
- Even Q’s are zero at x=0
- Even Q’s are \(-\infty\) at x=-1
- Odd Q’s are \( \infty \) at x=-1
- Number of zeros (times Q crosses x-axis) is equal to n+1.
Finally, the orthogonality condition is

\[
\int_{-1}^{1} P_n(z)P_m(z)dz = \begin{cases} 
0 & \text{for } n \neq m \\
\frac{2}{2n+1} & \text{for } n = m
\end{cases}
\]

In two dimensional spherical heat conduction, we normally do not worry about boundary conditions in the \( \theta \) direction. The primary condition to worry about is that the temperature must be finite throughout the entire domain. Hence, we normally throw away the Q Legendre polynomials.