When our differential equation takes the form
\[
\frac{d^2u}{dx^2} + \frac{1}{x} \frac{du}{dx} + \left[ \alpha^2 - \frac{n^2}{x^2} \right] u = 0
\]
we must recognize it as a Bessel equation with a solution given by
\[
u(x) = C_1 J_n(\alpha x) + C_1 Y_n(\alpha x)
\]
where
- \(J_n\): ordinary Bessel function of the first kind
- \(Y_n\): ordinary Bessel function of the second kind

The modified Bessel equation takes the form
\[
\frac{d^2u}{dx^2} + \frac{1}{x} \frac{du}{dx} - \left[ \alpha^2 - \frac{n^2}{x^2} \right] u = 0
\]
with a solution
\[
u(x) = C_1 I_n(\alpha x) + C_1 K_n(\alpha x)
\]
where
- \(I_n\): modified Bessel function of the first kind
- \(K_n\): modified Bessel function of the second kind

To better understand the nature of the Bessel functions, they are graphed below:
Some further properties
\[ J_0(0) = 1, \text{ other } J_n(0) = 0 \]
\[ Y_n(0) = -\infty \]
\[ I_0(0) = 1, \text{ other } I_n(0) = 0 \]
\[ K_n(0) = +\infty \]

Derivatives
\[ \frac{dJ_n(x)}{dx} = -\frac{1}{2} [J_{n-1}(x) - J_{n+1}(x)] \]
\[ \frac{dJ_0(x)}{dx} = -J_1(x) \]

which are valid for the Y Bessel function as well. For the modified Bessel functions
\[ \frac{dI_n(x)}{dx} = -\frac{1}{2} [I_{n-1}(x) + I_{n+1}(x)] \]
\[ \frac{dI_0(x)}{dx} = I_1(x) \]
\[ \frac{dK_n(x)}{dx} = -\frac{1}{2} [K_{n-1}(x) + K_{n+1}(x)] \]
\[ \frac{dK_0(x)}{dx} = -K_1(x) \]

The universal ordinary Bessel equation takes the form
\[ \frac{d^2u}{dx^2} + \left[ \frac{(1-2a)}{x} + 2\alpha \right] \frac{du}{dx} + \left[ \frac{(bc)^2 x^{(2c-2)}}{x} + \frac{\alpha(2a-1)}{x} + \frac{a^2 - n^2 c^2}{x^2} + \alpha^2 \right] u = 0 \]

which has a general solution of the form
\[ \theta(x) = x^a e^{\alpha x} \left[ C_1 J_n(bx^c) + C_2 Y_n(bx^c) \right] \]

The universal modified Bessel equation takes the form
\[ \frac{d^2u}{dx^2} + \left[ \frac{(1-2a)}{x} + 2\alpha \right] \frac{du}{dx} - \left[ \frac{(bc)^2 x^{(2c-2)}}{x} + \frac{\alpha(2a-1)}{x} + \frac{a^2 - n^2 c^2}{x^2} + \alpha^2 \right] u = 0 \]

which has a general solution of the form
\[ \theta(x) = x^a e^{\alpha x} \left[ C_1 I_n(bx^c) + C_2 K_n(bx^c) \right] \]

Here is a step by step procedure for taking an equation of the form
\[ \frac{d^2u}{dx^2} + f(x) \frac{du}{dx} + g(x) u = 0 \]

and mapping it into the universal solutions.

**Step 1**
For \( g(x) > 0 \), we have the solution to the universal ordinary Bessel equation. For \( g(x) < 0 \), we have the solution to the universal modified Bessel equation.

**Step 2**
We must have that \( f(x) \) takes the form
\[ f(x) = \frac{r_1}{x} + r_2 \]

Then
\[ \alpha = r_2/2 \]
\[ a = (1-r_1)/2 \]

**Step 3**
We must have that \( g(x) \) takes the form
\[ g(x) = s_1x^{s_2} + \frac{s_3}{x} + s_4x^{-2} + s_5 \]

**Step 4**
If \( \alpha = 0 \), then
\[ s_2 = 0, c=1, \text{ and } b = \sqrt{s_1} \]

Else if \( \alpha \neq 0 \) or \( s^5 \neq \alpha^2 \), then
\[ s_2 = 0, c=1, \text{ and } b = \sqrt{s_5 - \alpha^2} \]

Else if \( s_5 = \alpha^2 \) and \( s_3 = \alpha(2a-1) \)
\[ c = 1 + s_2/2 \text{ and } b = \sqrt{s_1} \]

Else if \( s_5 = \alpha^2 \), \( s_3 \neq \alpha(2a-1) \) and there are only \( 1/x \) and \( 1/x^2 \) terms
\[ c = -1 \text{ and } b = \sqrt{s_3} \]

Else if there is only a \( 1/x^2 \) term
\[ c = 0 \text{ and } b \text{ is indeterminate} \]

**Step 5**
Comparing the \( 1/x^2 \) terms gives
\[ n = \frac{\sqrt{a^2 - s_5}}{c} \]

**Step 6**
Write the general solution with \( \alpha, a, b, c, \) and \( n \) as found above.