ME 475
Spring 2011

Midterm 1

Note: This is a closed-book and closed-note examination. All questions must be answered in the space provided under each problem (use the back of the sheet, if necessary).

Pledge: I have neither given nor received any unauthorized assistance on this exam.

Signature: ________________________________

1. (40) The finite element equations for a linear two-noded bar element are:

\[
\frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}
\]

For the finite element mesh of bar elements shown below:

(a) Assemble the global finite element equations.
(b) Apply boundary conditions.
(c) Reduce the system of equations to a set that contains only unknown displacement DOFs (you do not have to solve for the nodal displacements).

Young's modulus: 4E7 psi
Cross-sectional area: 3 square inches

P = 20,000 lb
q = 1,000 lb/in (uniformly distributed)
(a) \( [B] = \begin{bmatrix} 4 & 2 \\ 3 & 1 \\ 1 & 4 \end{bmatrix} \)

\( K_1 = \frac{(4E7)(3)}{6} = 2E7 \)
\( K_2 = \frac{(4E7)(3)}{6} = 2E7 \)
\( K_3 = \frac{(4E7)(3)}{4} = 3E7 \)

**Global Equations:**

\[
\begin{bmatrix}
2+3 & 0 & -2 & -3 \\
0 & 2 & 0 & -2 \\
-2 & 0 & 2 & 0 \\
-3 & -2 & 0 & (2+3)
\end{bmatrix}
\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} =
\begin{bmatrix} F_1 + 2,000 \\ F_2 + 0 \\ F_3 + 0 \\ F_4 + 2,000 \end{bmatrix}
\]

(b) \( F_1 = 0 \)
\( U_2 = 0 \)
\( F_3 = -20,000/16 \)
\( F_4 = 0 \)

(c) **Reduced System:**

\[
\begin{bmatrix} 5 & -2 & -3 \\
-2 & 2 & 0 \\
-3 & 0 & 5 \end{bmatrix}
\begin{bmatrix} U_1 \\ U_3 \\ U_4 \end{bmatrix} =
\begin{bmatrix} 2,000 \\ -20,000 \\ 2,000 \end{bmatrix}
\]
2. (12) Consider the global finite element stiffness matrix shown below. Concerning the properties of this matrix before boundary conditions are applied, answer true or false to each of the following. This matrix is:

(a) sparse \[\boxed{T}\]
(b) singular \[\boxed{T}\]
(c) symmetric \[\boxed{T}\]
(d) square \[\boxed{T}\]
(e) positive definite \[\boxed{F}\]
(f) banded \[\boxed{F}\]

\[
\begin{bmatrix}
6 & 0 & 0 & -3 & 0 & 0 & -3 & 0 & 0 \\
0 & 5 & 0 & 0 & 0 & -1 & 0 & 0 & -4 \\
0 & 0 & 2 & 0 & 0 & -2 & 0 & 0 & 0 \\
-3 & 0 & 0 & 7 & 0 & 0 & 0 & -4 & 0 \\
0 & 0 & 0 & 0 & 4 & 0 & -2 & 0 & -2 \\
0 & -1 & -2 & 0 & 0 & 3 & 0 & 0 & 0 \\
-3 & 0 & 0 & 0 & -2 & 0 & 5 & 0 & 0 \\
0 & 0 & 0 & -4 & 0 & 0 & 0 & 4 & 0 \\
0 & -4 & 0 & 0 & -2 & 0 & 0 & 0 & 6
\end{bmatrix}
\begin{bmatrix}
U_1 \\
U_2 \\
U_3 \\
U_4 \\
U_5 \\
U_6 \\
U_7 \\
U_8 \\
U_9
\end{bmatrix}
= \begin{bmatrix}
F_1 \\
F_2 \\
F_3 \\
F_4 \\
F_5 \\
F_6 \\
F_7 \\
F_8 \\
F_9
\end{bmatrix}
\]

3. (8) What steps must be taken to make the above system of equations solvable?

\textit{Apply sufficient BC's to eliminate rigid body motion.}
4. (10) State the Principle of Total Potential Energy (PTPE).

FROM NOTES
5. (30) (a) Derive the interpolation functions $N_1$ and $N_2$ for a linear 2-noded bar element as shown below. Show all work.

(b) In the element above, if the two nodal displacements are known to be $u_1 = 5$ and $u_2 = 12$, what is the elemental displacement at the point $x = 3$?

(a) $u = a_0 + a_1 x$

$u(0) = u_1 = a_0$

$u(h) = u_2 = a_0 + a_1 h$

$\Rightarrow a_1 = \frac{u_2 - u_1}{h}$

$u = u_1 + \frac{u_2 - u_1}{h} x$

$= u_1 \left(1 - \frac{x}{h}\right) + u_2 \left(\frac{x}{h}\right)$

$= u_1 N_1 + u_2 N_2$

(b) $u(3) = 5 \left(1 - \frac{3}{8}\right) + 12 \left(\frac{3}{8}\right)$

$= 7.625$