ME451: Control Systems

Lecture 10
Linearization

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Course roadmap

Modeling
- Laplace transform
- Transfer function
- Models for systems
electrical
mechanical
electromechanical
- Block diagrams
- Linearization

Analysis
- Time response
  - Transient
  - Steady state
- Frequency response
  - Bode plot
- Stability
  - Routh-Hurwitz
  - Nyquist

Design
- Design specs
- Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

(Matlab simulations & laboratories)

Linear systems
- Easier to understand and obtain solutions
- Linear ordinary differential equations (ODEs),
  Homogeneous solution and particular solution
  Transient solution and steady state solution
  Solution caused by initial values, and forced solution
- Add many simple solutions to get more complex ones (use superposition!)
- Easy to check the Stability of stationary states (Laplace Transform)

Which is Linear
- Which of these equations is Linear
  They must obey superposition, test them
  \( y = 5x \)
  \( y = 5x + 3 \)
  \( y = 5x^2 + 3x \)
  \( y = 5\cos(x) \)

Why linearization?
- Real systems are inherently nonlinear. (Linear systems do not exist!) Ex. \( f(t) = Kx(t), v(t) = Ri(t) \)
- TF models are only for linear time-invariant (LTI) systems.
- Many control analysis/design techniques are available for linear systems.
- Nonlinear systems are difficult to deal with mathematically.
- Often we linearize nonlinear systems before analysis and design. How?

What is a linear system?
- A system having Principle of Superposition

A nonlinear system does not satisfy the principle of superposition.
**Six Steps to Linearization**

1. Identify the system model’s inputs and outputs
2. Express the model in the form $f(r, \dot{r}, \ddot{r}, ..., \epsilon, \dot{\epsilon}, \ddot{\epsilon}, ...) = 0$
3. Find an equilibrium where all input and output derivatives are zero and the original model is satisfied at the operating point $(r_o, \epsilon_o)$
   
   $f(r, \dot{r}, \ddot{r}, ..., \epsilon, \dot{\epsilon}, \ddot{\epsilon}, ...) = f(r, 0, 0, ..., \epsilon, 0, 0, ...) = 0$
   
   That is … Solve for $\epsilon_o$

4. Perform a Taylor Series expansion about the operating point retaining only 1st derivative terms.
   
   $f(r, \dot{r}, \ddot{r}, ..., \epsilon, \dot{\epsilon}, \ddot{\epsilon}, ...) = f(r, 0, 0, ..., \epsilon, 0, 0, ...) + \frac{\partial f}{\partial \epsilon} |_{\epsilon=0} \epsilon + \frac{\partial^2 f}{\partial \epsilon^2} |_{\epsilon=0} \epsilon^2 + \cdots$

5. Change variables
   
   $\hat{r} = (r-r_o), \hat{\dot{r}} = (\dot{r}-0), \hat{\ddot{r}} = (\ddot{r}-0), \ldots$
   
   $\hat{\epsilon} = (\epsilon-\epsilon_o), \hat{\dot{\epsilon}} = (\dot{\epsilon}-0), \hat{\ddot{\epsilon}} = (\ddot{\epsilon}-0), \ldots$

6. Rewrite the Taylor Expansion in the new variables
   
   $f(\hat{r}, \hat{\dot{r}}, \hat{\ddot{r}}, ..., \hat{\epsilon}, \hat{\dot{\epsilon}}, \hat{\ddot{\epsilon}}, ...) = f(r_o, 0, 0, ..., \epsilon_o, 0, 0, ...) + \frac{\partial f}{\partial \epsilon} |_{\epsilon=0} \epsilon + \frac{\partial^2 f}{\partial \epsilon^2} |_{\epsilon=0} \epsilon^2 + \cdots$

**Linearization**

- Nonlinear system: $\dot{x} = f(x, u)$
- Let $u_0$ be a nominal input and let the resultant state be $x_0$
- Perturbation: $u(t) = u_0(t) + \delta u(t)$
- Resultant perturb: $x(t) = x_0(t) + \delta x(t)$
- Taylor series expansion:
   
   $f(x, u) = f(x_0, u_0) + \frac{\partial f}{\partial x} |_{x=x_0, u=u_0} \delta x + \frac{\partial f}{\partial u} |_{x=x_0, u=u_0} \delta u + \mathcal{O}(\delta x^2, \delta u^2)$

**Linearization of a pendulum model**

- Motion of the pendulum
  
  $mL^2\ddot{\theta} + mgL\sin(\theta) = u(t)$

  \[ \dot{\theta}(t) + \frac{g}{L} \sin(\theta(t)) - \frac{u(t)}{mL^2} = 0 \]

- Linearize it at $\theta_0 = \pi$ (Defines output at O.P.)

- Find $u_0$ (Nonlinear System)
  
  $\dot{\theta} = \theta_0 + \delta \theta = \pi + \delta \theta$

  $u = u_0 + \delta u = 0 + \delta u$
Linearization of a pendulum model (cont')

- **Taylor series expansion**
  \[ f(\theta, \omega, u) = 0 \text{ at } (\theta_0, \omega_0) = (\pi, 0) \]

\[ \frac{\partial f(\theta, \omega, u)}{\partial \theta} \bigg|_{(\theta_0, \omega_0, u_0)} = g \cos \theta - \frac{g}{L} \theta_0 = \frac{1}{mL^2} \]

\[ \frac{\partial f(\theta, \omega, u)}{\partial \omega} \bigg|_{(\theta_0, \omega_0, u_0)} = \frac{1}{mL^2} \]

\[ f(\theta, \omega, u) = 0 \iff \frac{d}{d\theta} (\theta - \theta_0) + \frac{d}{du} (\theta - \theta_0) \bigg|_{(\theta_0, \omega_0, u_0)} \]

- **Change Variables**
  \[ \delta \theta = (\theta - \theta_0) \quad \delta \omega = (\omega - \omega_0) \quad \delta u = (u - u_0) \]

\[ \frac{\delta \ddot{\theta}}{L} - \frac{g}{mL^2} \delta \omega - \frac{1}{mL^2} \delta u = C \]

Summary and Exercises

- **Modeling of**
  - Nonlinear systems with linearization

- **Next**
  - More linearization examples