This lab is about modal analysis for a 3-mode system and for a variety of continuous systems.

**3-mode system review:**  \( F \) (only the 1st mass is directly forced)

Consider

Let's write down the equations:

**FBD:**

\[
 k_2 \left( x_2 - x_1 \right) \rightarrow m_2 \rightarrow k_3 \left( x_3 - x_2 \right)
\]

So,

\[
m_2 \ddot{x}_2 = -k_2 \left( x_2 - x_1 \right) + k_3 \left( x_3 - x_2 \right)
\]
\[ m_2 \dddot{x}_2 = -k_2 x_2 + k_2 x_1 + k_3 x_3 - k_3 x_2 \]

\[ m_2 \dddot{x}_2 + \begin{bmatrix} -k_2 & k_2 + k_3 & -k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \]

If we combine all 3 equations and write it in matrix form,

we get:

\[
\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} \dddot{x}_1 \\ \dddot{x}_2 \\ \dddot{x}_3 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} F \\ 0 \\ 0 \end{bmatrix}
\]

In the lab, we will examine a system where

\[ m = m_1 = m_2 = m_3 \quad \text{and} \quad k_1 = k_2 = k_3 = k_4 = k \]

So

\[
M = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \quad K = \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix}
\]
we are interested in the modal frequencies. Recall that we let: 
\[ x = \tilde{A} \cos(wt) \]
\[ \dot{x} = -\tilde{A} w \sin(wt) \]
then \[ \ddot{x} = -\tilde{A} w^2 \cos(wt) \]

Free Vibration

So
\[ M \left( -\tilde{A} w^2 \cos(wt) \right) + K \left( \tilde{A} \cos(wt) \right) = 0 \]

\[ \implies (-Mw^2 + K)\tilde{A} = 0. \]

Since \( \tilde{A} \neq 0 \), then \( \text{Det} \left[ -Mw^2 + K \right] = 0 \)

\[
\begin{vmatrix}
-mw^2 + 2K & -K & 0 \\
-K & -mw^2 + 2K & -K \\
0 & -K & -mw^2 + 2K
\end{vmatrix}
\]
\[-m w^2 + 2k \left[ ( -m w^2 + 2k )^2 - k^2 \right] - k \left( + ( +k)( -m w^2 + 2k ) \right) = 0 \]

\[\Rightarrow ( -m w^2 + 2k ) \left( ( -m w^2 + 2k )^2 - 2k^2 \right) = 0\]

This gives us \[w^2 = \frac{2k}{m} \Rightarrow w = \pm \sqrt{\frac{2k}{m}} = \sqrt{2} \sqrt{\frac{k}{m}}\]

and \[(-m w^2 + 2k)^2 = 2k^2\]

\[\Rightarrow -m w^2 + 2k = \pm k \sqrt{2}\]

\[\Rightarrow w^2 = -\frac{1}{m} \left( \pm k \sqrt{2} - 2k \right)\]

So \[w^2 = (2 + \sqrt{2}) \frac{k}{m}\]

then \[w = \sqrt{2 + \sqrt{2}} \sqrt{\frac{k}{m}}\]

since we like to order modal frequencies from least to greatest and \(> 0\),

\[\omega_1 = \sqrt{2 - \sqrt{2}} \sqrt{\frac{k}{m}}\]

\[\omega_2 = \sqrt{2} \sqrt{\frac{k}{m}}\]

\[\omega_3 = \sqrt{2 + \sqrt{2}} \sqrt{\frac{k}{m}}\]
As such, even if we aren't too sure about exactly $m$ and $k$, 

$$\frac{\omega_3}{\omega_2} = \frac{\sqrt{2+\sqrt{3}}}{\sqrt{2}} \approx 1.31$$

$$\frac{\omega_2}{\omega_1} = \frac{\sqrt{2}}{\sqrt{2-\sqrt{3}}} \approx 1.84$$

In the lab, the masses and spring coefficient are approx.

$m = \text{block mass} + \text{carriage mass} \approx 500 \text{g} + 700 \text{g} = 1.2 \text{ kg}$

and $K \approx 450 \text{ N/m}$

then \( \sqrt{\frac{K}{m}} = \sqrt{\frac{450}{1.2}} \approx 19.36 \frac{\text{rad}}{\text{sec}} \approx 3.08 \text{ Hz} \)

As a result, we might expect

\( f_1 = 2.36 \text{ Hz} \)
\( f_2 = 4.35 \text{ Hz} \)
\( f_3 = 5.19 \text{ Hz} \)
Now, each of these “eigenfrequencies” correspond to “eigenmodes,” or mode shapes. Although this can be done analytically by finding $\ddot{A}_i$ that satisfies $[-\omega_i^2 \bar{M} + \bar{K}] \ddot{A}_i = 0$.

Since the forcing is only on mass 1, it will excite different modes with a different amount of force.

**Continuous System Review**

Instead of a finite # of modes, a continuous system will have an infinite # of modes.

\[
\text{Eg) } \quad \text{A cantilever beam}
\]
Although we won't get into the math of how to find the modal frequencies or mode shapes, we can experimentally see them using a frequency sweep or impulse response. To observe the response, we will use

- Encoders (for rotary displacement)
- Our visual response
- Piezo electric sensors

\[
\text{piezo patch} \rightarrow R \rightarrow V \rightarrow \text{we measure the voltage output from the piezo circuit.}
\]