ME451: Control Systems

Lecture 9

Stability

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Course roadmap

Modeling

- ✓ Laplace transform
- ✓ Transfer function
- ✓ Models for systems
  - ✓ electrical
  - ✓ mechanical
  - ✓ electromechanical
- ✓ Block diagrams
- ✓ Linearization

Analysis

- Time response
  - Transient
  - Steady state
- Frequency response
  - Bode plot
- ➤ Stability
  - Routh-Hurwitz
  - (Nyquist)

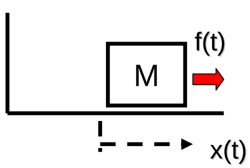
Design

- Design specs
- Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

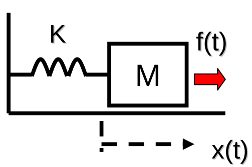
(Matlab simulations &) laboratories

Simple mechanical examples

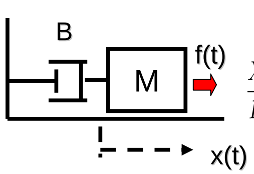
- We want mass to stay at x=0, but wind gave some initial speed (F(t)=0). What will happen?



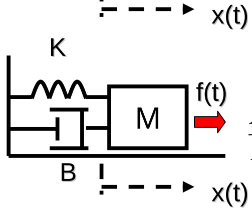
$$\frac{X(s)}{F(s)} = \frac{1}{s^2}$$



$$\frac{X(s)}{F(s)} = \frac{1}{s^2 + K}$$



$$\frac{X(s)}{F(s)} = \frac{1}{s^2 + Bs}$$



$$\frac{X(s)}{F(s)} = \frac{1}{s^2 + Bs + K}$$

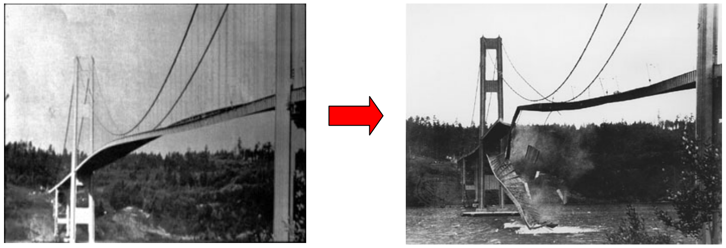
- How to characterize different behaviors with TF?

Stability

- Utmost important specification in control design!
- Unstable systems have to be stabilized by feedback.
- Unstable closed-loop systems are useless.
  - What happens if a system is unstable?
    - may hit mechanical/electrical “stops” (saturation)
    - may break down or burn out

What happens if a system is unstable?

Tacoma Narrows Bridge (July 1-Nov.7, 1940)

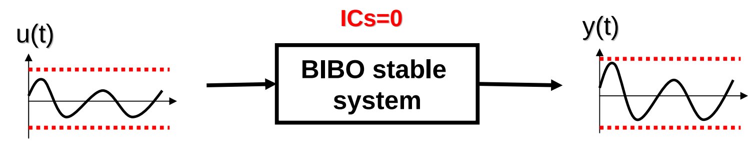


Wind-induced vibration　　Collapsed!

2008…

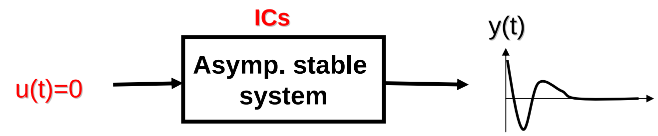
Mathematical definitions of stability

- **BIBO** (Bounded-Input-Bounded-Output) **stability** :
  Any bounded input generates a bounded output.



- **Asymptotic stability** :

  Any ICs generates y(t) converging to zero.



Some terminologies

$$G(s) = \frac{n(s)}{d(s)}$$

Ex. $$G(s) = \frac{(s-1)(s+1)}{(s+2)(s^2+1)}$$

- *Zero* : roots of n(s)　　(Zeros of G) $= \pm 1$
- *Pole* : roots of d(s)　　(Poles of G) $= -2, \pm j$
- *Characteristic polynomial* : d(s)
- *Characteristic equation* : d(s)=0

Stability condition in s-domain

(Proof omitted, and not required)

For a system represented by a transfer function G(s),

system is BIBO stable

⇕

All the poles of G(s) are in the open left half of the complex plane.

⇕

system is asymptotically stable

"Idea" of stability condition

Example $y'(t) + \alpha y(t) = u(t),\ y(0) = y_0$

$\Rightarrow$ $sY(s) - y(0) + \alpha Y(s) = U(s)$

$\Rightarrow$ $Y(s) = \dfrac{1}{s+\alpha}(U(s) + y(0))$

Asym. Stability: (U(s)=0) $y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s+\alpha}y(0)\right\} = e^{-\alpha t}y(0) \to 0 \Leftrightarrow \mathrm{Re}(\alpha) > 0$

BIBO Stability: (y(0)=0) $y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{G(s)U(s)\} = \int_0^t g(\tau)u(t-\tau)d\tau = \int_0^t e^{-\alpha\tau}u(t-\tau)d\tau$

$$|y(t)| \le \int_0^t |e^{-\alpha\tau}||u(t-\tau)|d\tau \le \underline{\int_0^t |e^{-\alpha\tau}|d\tau} \cdot u_{max}$$

Bounded if Re(α)>0

Second order impulse response- Underdamped and Undamped

$$\delta(t) \longrightarrow \boxed{\dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}} \longrightarrow h(t)$$

$$p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$\zeta > 1$ … Overdamped

$\zeta = 1$ … Critically damped

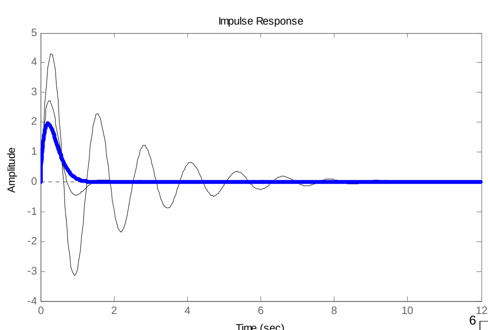
$1 > \zeta > 0$ … Underdamped

$\zeta = 0$ … Undamped

$$h(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}t)1(t)$$

Second order impulse response – Underdamped and Undamped

Changing ζ / Fixed ω_n



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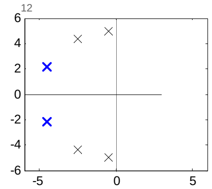
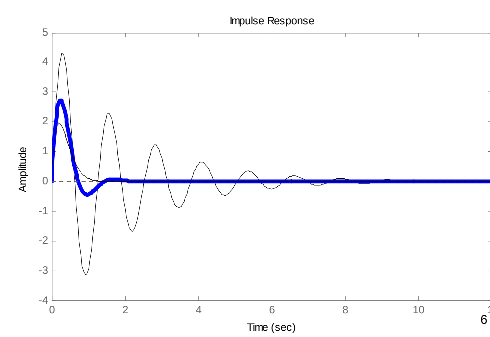


Second order impulse response – Underdamped and Undamped

Changing ζ / Fixed ω_n



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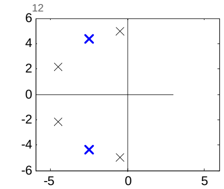
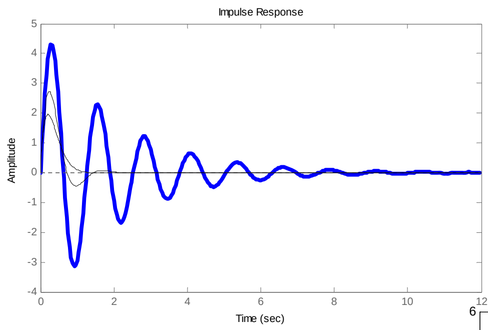
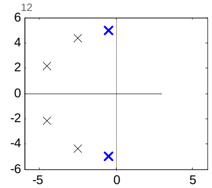


Second order impulse response – Underdamped and Undamped

Changing ζ / Fixed ω_n





$$h(t)=\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}t)1(t)$$

Remarks on stability

- For a general system (nonlinear etc.), BIBO stability condition and asymptotic stability condition are different.
- For linear time-invariant (LTI) systems (to which we can use Laplace transform and we can obtain a transfer function), the conditions happen to be the same.
- In this course, we are interested in only LTI systems, we use simply "stable" to mean both BIBO and asymptotic stability.

Remarks on stability (cont'd)

- Marginally stable if
  - G(s) has no pole in the open RHP (Right Half Plane), &
  - G(s) has at least one simple pole on $j\omega$-axis, &
  - G(s) has no multiple poles on $j\omega$-axis.

$$G(s)=\frac{1}{s(s^2+4)(s+1)} \qquad G(s)=\frac{1}{s(s^2+4)^2(s+1)}$$

Marginally stable — ***NOT marginally stable***

- Unstable if a system is neither stable nor marginally stable.

Examples

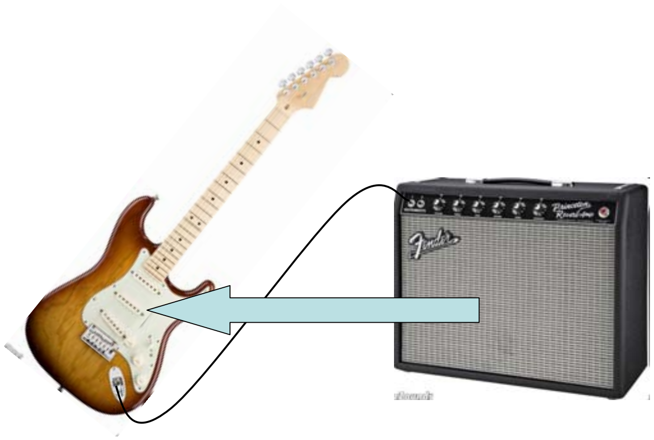
- Repeated poles

$$\mathcal{L}^{-1}\left[\frac{2\omega s}{(s^2+\omega^2)^2}\right]=t\sin\omega t \qquad \mathcal{L}^{-1}\left[\frac{s^2-\omega^2}{(s^2+\omega^2)^2}\right]=t\cos\omega t$$

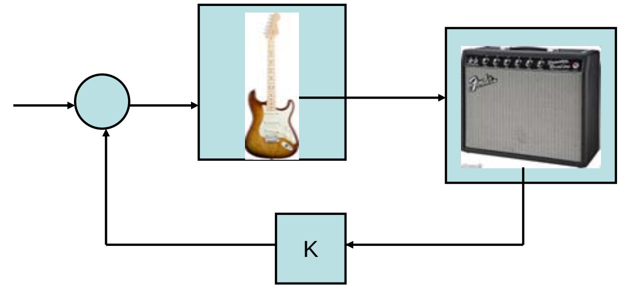
$$\cdots=t^2\sin\omega t \qquad \cdots=t^2\cos\omega t$$

- Does marginal stability imply BIBO stability?
  - TF: $G(s)=\frac{2s}{(s^2+1)}$
  - Pick $u(t)=\sin t \;\overrightarrow{\mathcal{L}}\; U(s)=\frac{1}{(s^2+1)}$
  - Output $\mathcal{L}^{-1}\left[Y(s)=G(s)U(s)=\frac{2s}{(s^2+1)^2}\right]=t\sin t$

Feedback Technique



Positive Feedback



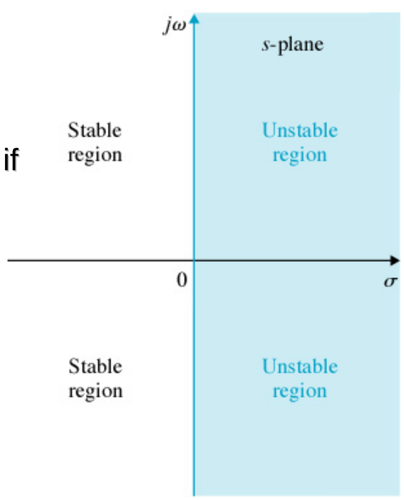
K will depends on the distance between the guitar and the amplifier.

Stability summary

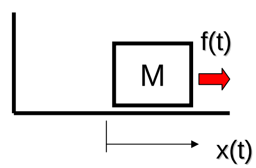
Let s_i be poles of G.
Then, G is …

- (BIBO, asymptotically) stable if $\text{Re}(s_i)<0$ for all i.
- marginally stable if
  - $\text{Re}(s_i)<=0$ for all i, and
  - simple root for $\text{Re}(s_i)=0$
- unstable if

  it is neither stable nor marginally stable.

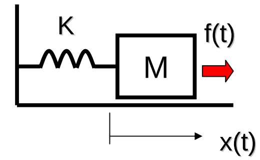


Mechanical examples: revisited



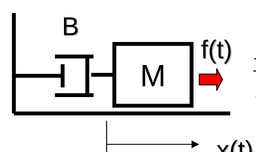
$$\frac{X(s)}{F(s)} = \frac{1}{s^2}$$

Poles=
stable?



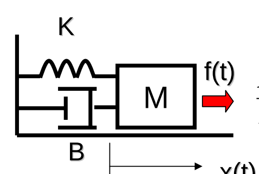
$$\frac{X(s)}{F(s)} = \frac{1}{s^2+K}$$

Poles=
stable?



$$\frac{X(s)}{F(s)} = \frac{1}{s^2+Bs}$$

Poles=
stable?



$$\frac{X(s)}{F(s)} = \frac{1}{s^2+Bs+K}$$

Poles=
stable?

Examples

$G(s)$	Stable/marginally stable /unstable
$\frac{5(s+2)}{(s+1)(s^2+s+1)}$	?
$\frac{5(-s+2)}{(s+1)(s^2+s+1)}$	?
$\frac{5}{(s-2)(s^2+3)}$	?
$\frac{s^2+3}{(s+1)(s^2-s+1)}$	?
$\frac{1}{(s+1)(s^2+1)^2}$	?
$\frac{1}{(s^2-1)(s+1)}$	???

Summary and Exercises

- Stability for LTI systems
  - (BIBO and asymptotically) stable, marginally stable, unstable
  - Stability for G(s) is determined by poles of G.
- Next
  - Routh-Hurwitz stability criterion to determine stability without explicitly computing the poles of a system.
- Exercises
  - Solve examples in the previous slide.