ME451: Control Systems

Lecture 5
Modeling of mechanical systems

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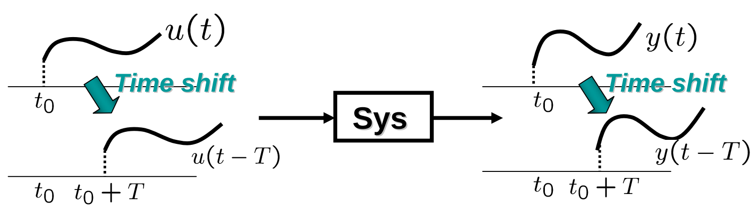
Course roadmap

Modeling	*Analysis*	*Design*
✓ Laplace transform ✓ Transfer function ✓ Models for systems ✓ electrical • mechanical • electromechanical Block diagrams Linearization	Time response • Transient • Steady state Frequency response • Bode plot Stability • Routh-Hurwitz • Nyquist	Design specs Root locus Frequency domain PID & Lead-lag Design examples

(Matlab simulations &) laboratories

Time-invariant & time-varying

- A system is called ***time-invariant (time-varying)*** if system parameters do not (do) change in time.
- Example: Mx''(t)=f(t) & M(t)x''(t)=f(t)
- For time-invariant systems:

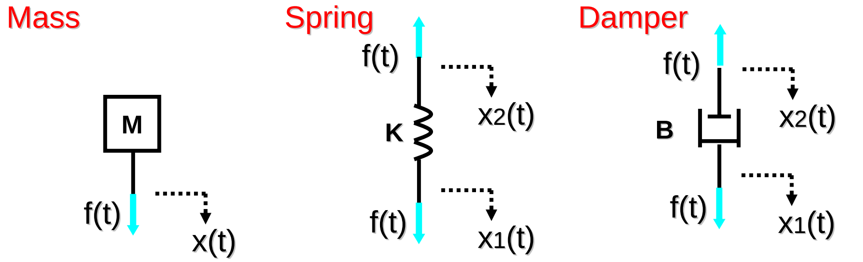


- This course deals with time-invariant systems.

Newton's laws of motion

- 1st law:
  - A particle remains at rest or continues to move in a straight line with a constant velocity if there is no unbalancing force acting on it.
- 2nd law:
  - $\sum F_i(t) = m\frac{d^2x}{dt^2}$: translational
  - $\sum \tau_i(t) = I\frac{d^2\theta}{dt^2}$: rotational
- 3rd law:
  - For every action has an equal and opposite reaction

Translational mechanical elements: (constitutive equations)

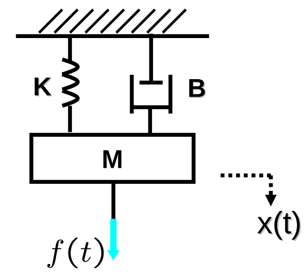


$$f(t) = Mx''(t) \qquad f(t) = K(x_1(t) - x_2(t)) \qquad f(t) = B(x_1'(t) - x_2'(t))$$

$$\begin{pmatrix} x(0) = 0 \\ \dot{x}(0) = 0 \end{pmatrix} \qquad \begin{pmatrix} x_1(0) = 0 \\ x_2(0) = 0 \end{pmatrix}$$

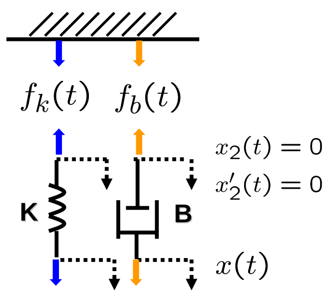
$$F(s) = Ms^2X(s) \qquad F(s) = K(X_1(s) - X_2(s)) \qquad F(s) = Bs(X_1(s) - X_2(s))$$

Mass-spring-damper system



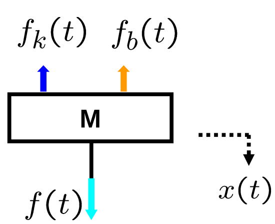
$$Mx''(t) + Bx'(t) + Kx(t) = f(t)$$

Free body diagram



$f_k(t) = K(x(t) - 0)$ $\quad f_b(t) = B(x'(t) - 0)$

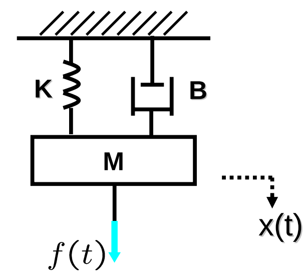
Direction of actual force will be automatically determined by the relative values!



- Newton's law: F=ma

$$Mx''(t) = f(t) - f_k(t) - f_b(t) = f(t) - Kx(t) - Bx'(t)$$

Mass-spring-damper system



- Equation of motion

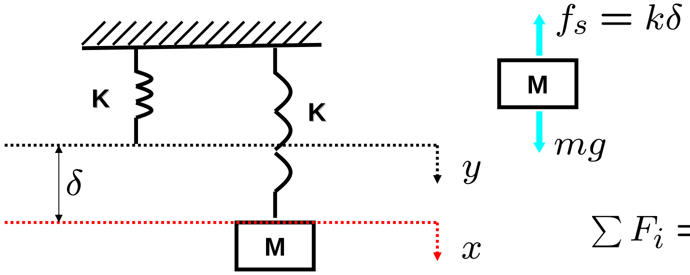
$$Mx''(t) + Bx'(t) + Kx(t) = f(t)$$

- By Laplace transform (with zero initial conditions),

$$X(s) = \frac{1}{Ms^2 + Bs + K}F(s)$$

(2nd order system)

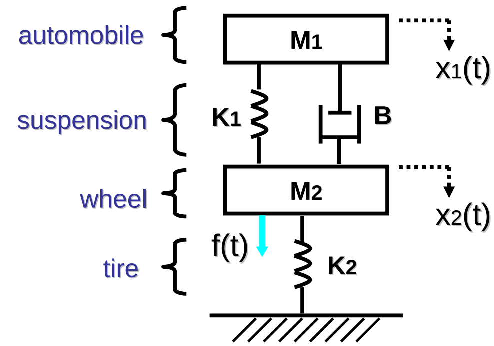
Gravity?



$$\sum F_i = ma$$

- At rest, $\sum F_i = -k\delta + mg = 0$
- y coordinate: $m\ddot{y} = mg - ky$
- x coordinate: $m\ddot{x} = mg - k(x+\delta)$ $= -kx$

Automobile suspension system



$$\begin{cases} M_1 x_1''(t) &= -B(x_1'(t) - x_2'(t)) - K_1(x_1(t) - x_2(t)) \\ M_2 x_2''(t) &= f(t) - B(x_2'(t) - x_1'(t)) - K_1(x_2(t) - x_1(t)) - K_2 x_2(t) \end{cases}$$

Automobile suspension system

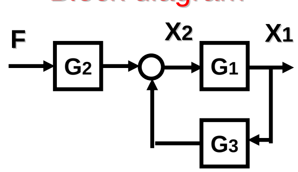
$$\begin{cases} M_1 x_1''(t) &= -B(x_1'(t) - x_2'(t)) - K_1(x_1(t) - x_2(t)) \\ M_2 x_2''(t) &= f(t) - B(x_2'(t) - x_1'(t)) - K_1(x_2(t) - x_1(t)) - K_2 x_2(t) \end{cases}$$

Laplace transform with zero ICs

$$\begin{cases} M_1 s^2 X_1(s) &= -B(sX_1(s) - sX_2(s)) - K_1(X_1(s) - X_2(s)) \\ M_2 s^2 X_2(s) &= F(s) - B(sX_2(s) - sX_1(s)) - K_1(X_2(s) - X_1(s)) - K_2 X_2(s) \end{cases}$$

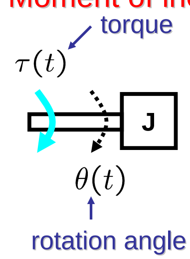
$$\begin{cases} X_1(s) &= \underbrace{\frac{Bs + K_1}{M_1 s^2 + Bs + K_1}}_{G_1(s)} X_2(s) \\ X_2(s) &= \underbrace{\frac{1}{M_2 s^2 + Bs + K_1 + K_2}}_{G_2(s)} F(s) + \underbrace{\frac{Bs + K_1}{M_2 s^2 + Bs + K_1 + K_2}}_{G_3(s)} X_1(s) \end{cases}$$

Block diagram

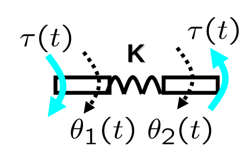


Rotational mechanical elements (constitutive equations)

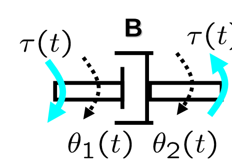
Moment of inertia



Rotational spring



Friction

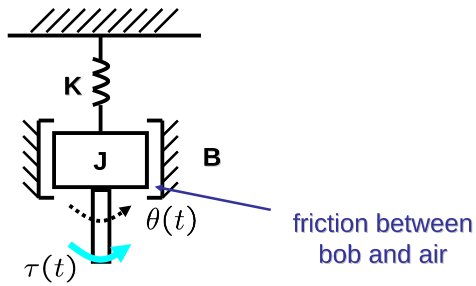


$$\tau(t) = J\theta''(t)$$
$$\begin{pmatrix} \theta(0) = 0 \\ \dot{\theta}(0) = 0 \end{pmatrix}$$
$$T(s) = Js^2\Theta(s)$$

$$\tau(t) = K(\theta_1(t) - \theta_2(t))$$
$$T(s) = K(\Theta_1(s) - \Theta_2(s))$$

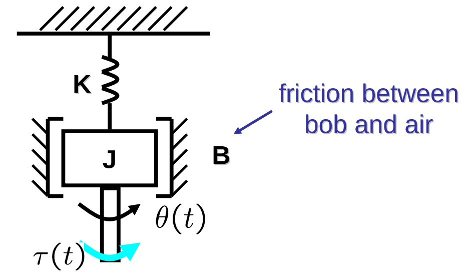
$$\tau(t) = B(\theta_1'(t) - \theta_2'(t))$$
$$\begin{pmatrix} \Theta_1(0) = 0 \\ \Theta_2(0) = 0 \end{pmatrix}$$
$$T(s) = Bs(\Theta_1(s) - \Theta_2(s))$$

Torsional pendulum system Ex.2.12



$$J\theta''(t) + B\theta'(t) + K\theta(t) = \tau(t)$$

Torsional pendulum system



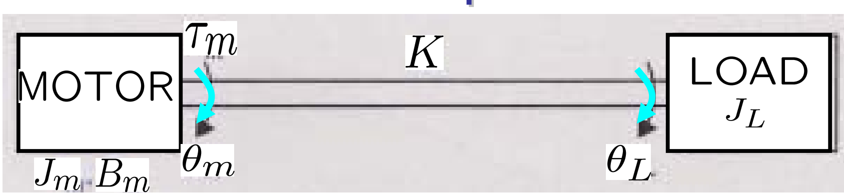
- Equation of Motion

$$J\theta''(t) + B\theta'(t) + K\theta(t) = \tau(t)$$

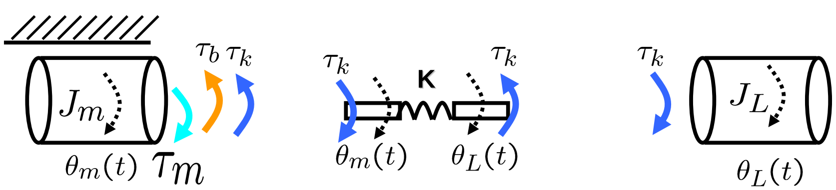
- By Laplace transform (with zero ICs),

$$G(s) = \frac{\Theta(s)}{T(s)} = \frac{1}{Js^2 + Bs + K}$$ *(2nd order system)*

Example

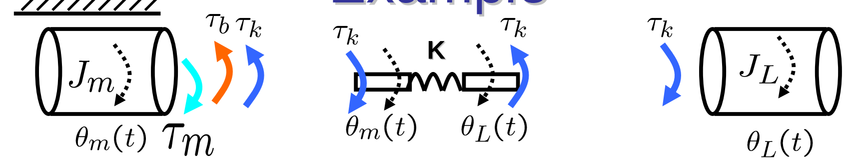


- FBD



$\tau_b = B_m(\dot{\theta}_m - 0)$ $\quad$ $\tau_k = K(\theta_m - \theta_L)$

Example



$\tau_b = B_m(\dot{\theta}_m - 0)$ $\quad$ $\tau_k = K(\theta_m - \theta_L)$

- By Newton's law

$$\begin{cases} J_m\theta_m''(t) &= \tau_m(t) - B_m\theta_m'(t) - K(\theta_m(t) - \theta_L(t)) \\ J_L\theta_L''(t) &= K(\theta_m(t) - \theta_L(t)) \end{cases}$$

- By Laplace transform (with zero ICs),

$$\begin{cases} J_m s^2\Theta_m(s) &= T_m(s) - B_m s\Theta_m(s) - K(\Theta_m(s) - \Theta_L(s)) \\ J_L s^2\Theta_L(s) &= K(\Theta_m(s) - \Theta_L(s)) \end{cases}$$

Example (cont'd)

- From second equation:

$$\Theta_L(s) = \underbrace{\frac{K}{J_L s^2 + K}}_{G_1(s)} \Theta_m(s)$$

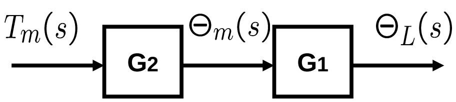
(2nd order system)

- From first equation:

$$\Theta_m(s) = \underbrace{\frac{J_L s^2 + K}{s\left(J_m J_L s^3 + B_m J_L s^2 + K(J_m + J_L)s + B_m K\right)}}_{G_2(s)} T_m(s)$$

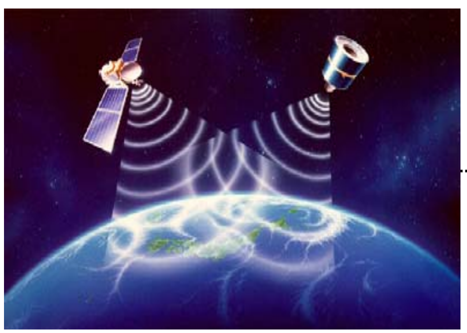
(4th order system)

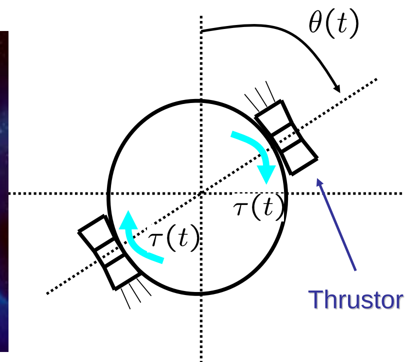
Block diagram



Satellite Picture

Rigid satellite Ex. 2.13





- Broadcasting
- Weather forecast
- Communication
- GPS, etc.

$$\tau(t) = J\ddot{\theta}(t)$$

$$\Rightarrow G(s) = \frac{\Theta(s)}{T(s)} = \frac{1}{Js^2}$$

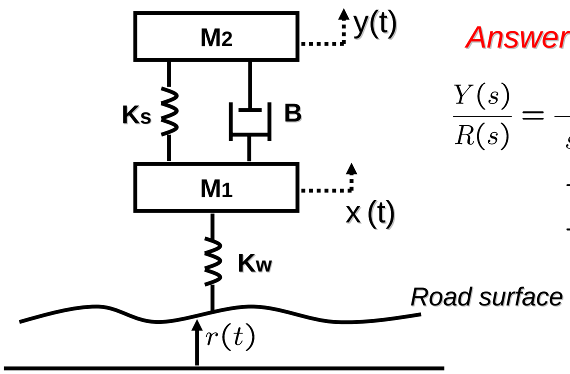
Double integrator

Summary & Exercises

- Modeling of mechanical systems
  - Translational
  - Rotational
- Next, block diagrams.
- Exercises
  - Derive equations for the automobile suspension problem.

Exercises (Franklin et al.)

- *Quarter car model*: Obtain a transfer function from R(s) to Y(s).



Answer

$$\frac{Y(s)}{R(s)} = \frac{\frac{k_w b}{m_1 m_2}\left(s + \frac{k_s}{b}\right)}{s^4 + \left(\frac{b}{m_1} + \frac{b}{m_2}\right)s^3 + \left(\frac{k_s}{m_1} + \frac{k_s}{m_2} + \frac{k_w}{m_1}\right)s^2 + \left(\frac{k_w b}{m_1 m_2}\right)s + \frac{k_w k_s}{m_1 m_2}}$$