ME451: Control Systems

Lecture 3
Solution to ODEs via Laplace transform

Dr. Jongeun Choi
Department of Mechanical Engineering
Michigan State University

Course roadmap

Modeling

- Laplace transform
- Transfer function
- Models for systems
  - electrical
  - mechanical
  - electromechanical
- Block diagrams
- Linearization

Analysis

- Time response
  - Transient
  - Steady state
- Frequency response
  - Bode plot
- Stability
  - Routh-Hurwitz
  - Nyquist

Design

- Design specs
- Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

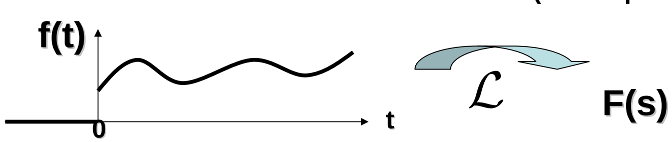
(Matlab simulations &) laboratories

Laplace transform (review)

- One of most important math tools in the course!
- Definition: For a function f(t) (f(t)=0 for t<0),

$$F(s) = \mathcal{L}\{f(t)\} := \int_0^{\infty} f(t)e^{-st}dt$$

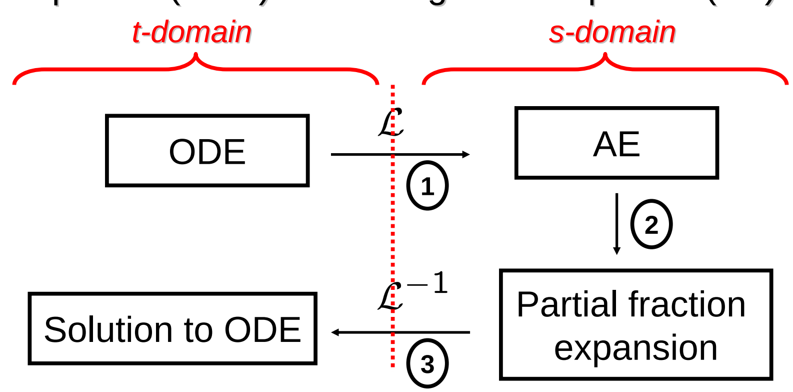
(s: complex variable)



- We denote Laplace transform of f(t) by F(s).

An advantage of Laplace transform

- We can transform an ordinary differential equation (ODE) into an algebraic equation (AE).



Example 1

ODE with initial conditions (ICs)

$$\frac{d^2y(t)}{dt^2}+3\frac{dy(t)}{dt}+2y(t)=5u_s(t),\ y(0)=-1,\ y'(0)=2$$

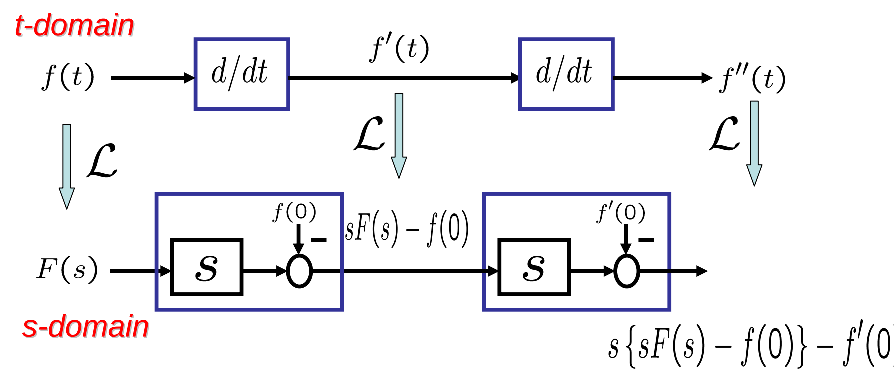
1. Laplace transform

$$\underbrace{s^2Y(s)-sy(0)-y'(0)}_{\mathcal{L}\{y''(t)\}}+3\underbrace{\{sY(s)-y(0)\}}_{\mathcal{L}\{y'(t)\}}+2Y(s)=\frac{5}{s}$$

$$\Rightarrow\quad Y(s)=\frac{-s^2-s+5}{s(s+1)(s+2)}$$

Properties of Laplace transform *Differentiation (review)*

$$\mathcal{L}\{f'(t)\}=sF(s)-f(0)$$



Example 1 (cont'd)

2. Partial fraction expansion

unknowns

$$Y(s)=\frac{-s^2-s+5}{s(s+1)(s+2)}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+2}$$

Multiply both sides by s & let s go to zero:

$$sY(s)|_{s\to 0}=A+s\frac{B}{s+1}\Big|_{s\to 0}+s\frac{C}{s+2}\Big|_{s\to 0}\quad\Rightarrow\quad A=sY(s)|_{s\to 0}=\frac{5}{2}$$

Similarly,

$$\begin{aligned}B&=(s+1)Y(s)|_{s\to -1}=\cdots=-5\\C&=(s+2)Y(s)|_{s\to -2}=\cdots=\frac{3}{2}\end{aligned}$$

Example 1 (cont'd)

3. Inverse Laplace transform

$$\mathcal{L}^{-1}\left\{Y(s)=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+2}\right\}$$

$$\Rightarrow\quad y(t)=\left(\underbrace{\frac{5}{2}}_{A}+\underbrace{(-5)}_{B}e^{-t}+\underbrace{\frac{3}{2}}_{C}e^{-2t}\right)u_s(t)$$

If we are interested in only the final value of y(t), apply Final Value Theorem:

$$\lim_{t\to\infty}y(t)=\lim_{s\to 0}sY(s)=\lim_{s\to 0}\frac{-s^2-s+5}{(s+1)(s+2)}=\frac{5}{2}$$

Example 2

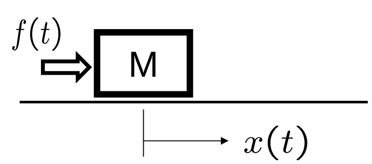
$$\ddot{y}(t) - y(t) = t, \quad y(0) = 1, \dot{y}(0) = 1$$

- S1 $s^2Y(s) - sy(0) - \dot{y}(0) - Y(s) = \frac{1}{s^2},$
- S2 $$Y(s) = \frac{1}{s-1} + \frac{1}{s^2(s^2-1)}$$
$$= \frac{1}{s-1} + \frac{s^2 - (s^2-1)}{s^2(s^2-1)} = \frac{1}{s-1} + \boxed{\frac{1}{s^2-1}} - \frac{1}{s^2}$$
$$= \frac{1}{s-1} + \boxed{\frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1}} - \frac{1}{s^2}$$
- S3 $y(t) = \mathcal{L}^{-1}(Y(s)) = \left[\frac{3}{2}e^t - \frac{1}{2}e^{-t} - t\right] u_s(t)$

In this way, we can find a rather complicated solution to ODEs easily by using Laplace transform table!

Example: Newton's law

$$M\frac{d^2x(t)}{dt^2} = f(t)$$



We want to know the trajectory of x(t). By Laplace transform,

$$M\left(s^2X(s) - sx(0) - x'(0)\right) = F(s)$$

$$\Rightarrow X(s) = \frac{1}{Ms^2}F(s) + \frac{x(0)}{s} + \frac{x'(0)}{s^2}$$

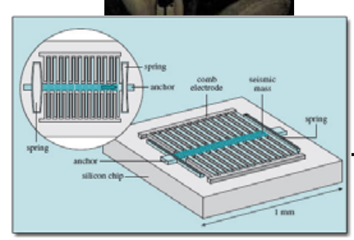
(Total response) = (Forced response) + (Initial condition response)

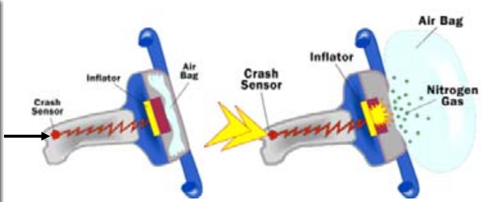
$$\Rightarrow x(t) = \mathcal{L}^{-1}\left[\frac{1}{Ms^2}F(s)\right] + x(0)u_s(t) + x'(0)tu_s(t)$$

EX. Air bag and accelerometer

- Tiny MEMS accelerometer
  - Microelectromechanical systems (MEMS)

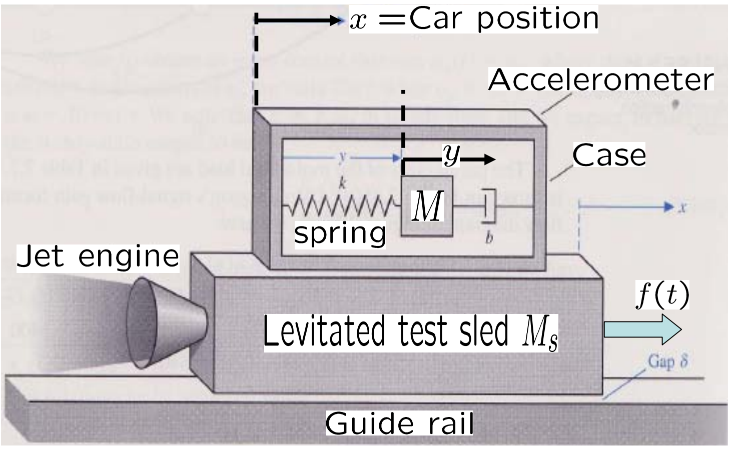






(Pictures from various websites)

Ex: Mechanical accelerometer



Ex: Mechanical accelerometer (cont'd)

- We would like to know how y(t) moves when unit step f(t) is applied with zero ICs.
- By Newton's law

$$\begin{cases} M\frac{d^2}{dt^2}(x(t)+y(t)) = -b\frac{dy(t)}{dt} - ky(t) \\ M_s\frac{d^2x(t)}{dt^2} = f(t) \end{cases}$$

$$\Rightarrow \quad My''(t)+by'(t)+ky(t) = -\frac{M}{M_s}f(t)$$

$$\overset{\mathcal{L}}{\Rightarrow} \quad Y(s) = -\frac{M}{M_s}\cdot\frac{1}{Ms^2+bs+k}\cdot\frac{1}{s} = -\frac{1}{M_s}\cdot\frac{1}{s^2+(b/M)s+(k/M)}\cdot\frac{1}{s}$$

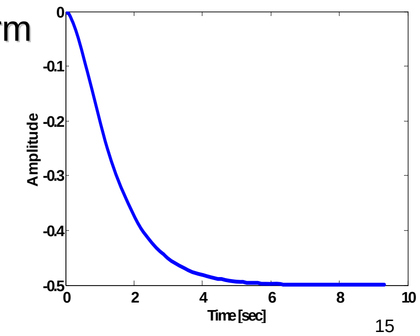
Ex: Mechanical accelerometer (cont'd)

- Suppose that b/M=3, k/M=2 and Ms=1.
- Partial fraction expansion

$$Y(s) = -\frac{1}{s^2+3s+2}\cdot\frac{1}{s} = -\frac{1}{2s}+\frac{1}{s+1}-\frac{1}{2(s+2)}$$

- Inverse Laplace transform

$$y(t) = \left(-\frac{1}{2}+e^{-t}-\frac{1}{2}e^{-2t}\right)u_s(t)$$



Summary & Exercises

- Solution procedure to ODEs
  1. Laplace transform
  2. Partial fraction expansion
  3. Inverse Laplace transform
- Next, modeling of physical systems using Laplace transform
- Exercises
  - Derive the solution to the accelerometer problem.
  - E2.4 of the textbook in page 135.