# ME451: Control Systems

## Lecture 3
## Solution to ODEs via Laplace transform


Dr. Jongeun Choi
Department of Mechanical Engineering
Michigan State University




# Course roadmap

***Modeling***

- Laplace transform
- Transfer function
- Models for systems
  - electrical
  - mechanical
  - electromechanical
- Linearization

***Analysis***

- Time response
  - Transient
  - Steady state
- Frequency response
  - Bode plot
- Stability
  - Routh-Hurwitz
  - Nyquist

***Design***

- Design specs
- Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

*(Matlab simulations &) laboratories*

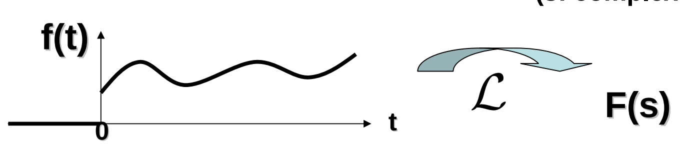


# Laplace transform (review)

- One of most important math tools in the course!
- Definition: For a function f(t) (f(t)=0 for t<0),

$$F(s) = \mathcal{L}\{f(t)\} := \int_0^\infty f(t)e^{-st}dt$$

**(s: complex variable)**

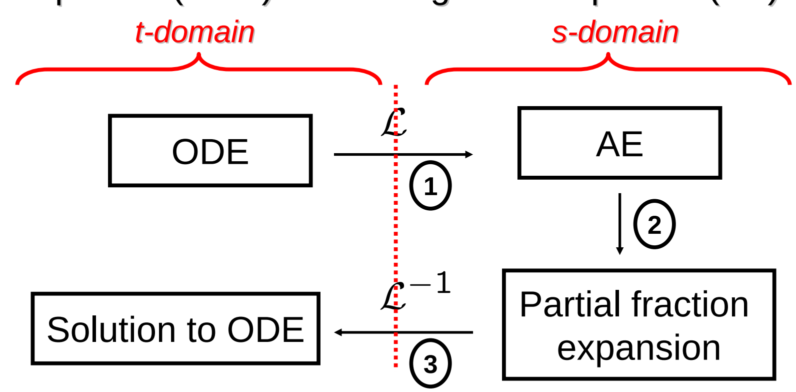



- We denote Laplace transform of f(t) by F(s).



# An advantage of Laplace transform

- We can transform an ordinary differential equation (ODE) into an algebraic equation (AE).



# Example 1

ODE with initial conditions (ICs)

$$\frac{d^2y(t)}{dt^2}+3\frac{dy(t)}{dt}+2y(t)=5u_s(t),\ y(0)=-1,\ y'(0)=2$$

1. Laplace transform

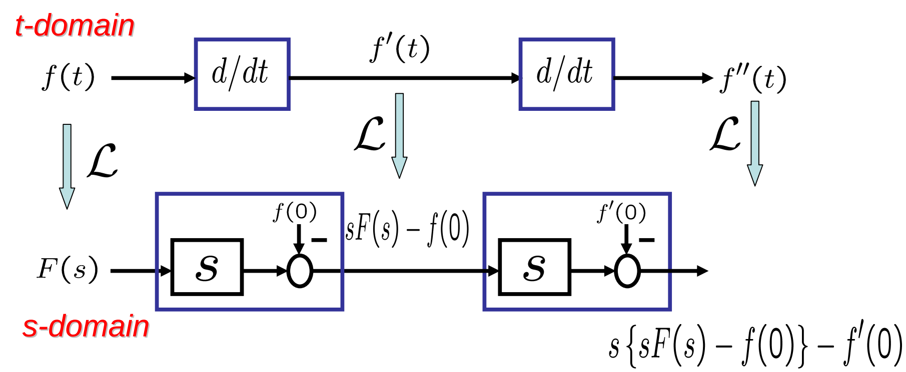
$$\underbrace{\qquad\qquad\qquad}_{\mathcal{L}\{y''(t)\}}\qquad\underbrace{\qquad\qquad}_{\mathcal{L}\{y'(t)\}}$$

$\Rightarrow$



# Properties of Laplace transform *Differentiation (review)*

$$\mathcal{L}\{f'(t)\}=sF(s)-f(0)$$






# Example 1 (cont'd)

2. Partial fraction expansion

unknowns

$$Y(s)=\frac{-s^2-s+5}{s(s+1)(s+2)}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+2}$$

Multiply both sides by s & let s go to zero:

$$sY(s)|_{s\to 0}=A+s\frac{B}{s+1}\Big|_{s\to 0}+s\frac{C}{s+2}\Big|_{s\to 0}\ \Rightarrow$$

Similarly,



# Example 1 (cont'd)

3. Inverse Laplace transform

$$\mathcal{L}^{-1}\left\{Y(s)=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+2}\right\}$$

$\Rightarrow$

If we are interested in only the final value of y(t), apply Final Value Theorem:

$$\lim_{t\to\infty}y(t)=\lim_{s\to 0}sY(s)=\lim_{s\to 0}\frac{-s^2-s+5}{(s+1)(s+2)}=\frac{5}{2}$$

# Example 2

$$\ddot{y}(t) - y(t) = t, \quad y(0) = 1, \dot{y}(0) = 1$$

- S1
- S2
- S3 $y(t) = \mathcal{L}^{-1}(Y(s)) =$

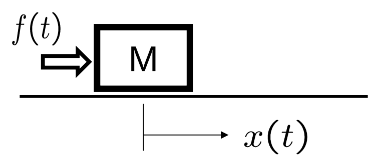


**In this way, we can find a rather complicated solution to ODEs easily by using Laplace transform table!**



# Example: Newton's law

$$M\frac{d^2x(t)}{dt^2} = f(t)$$




We want to know the trajectory of x(t). By Laplace transform,

$$M\left(s^2X(s) - sx(0) - x'(0)\right) = F(s)$$

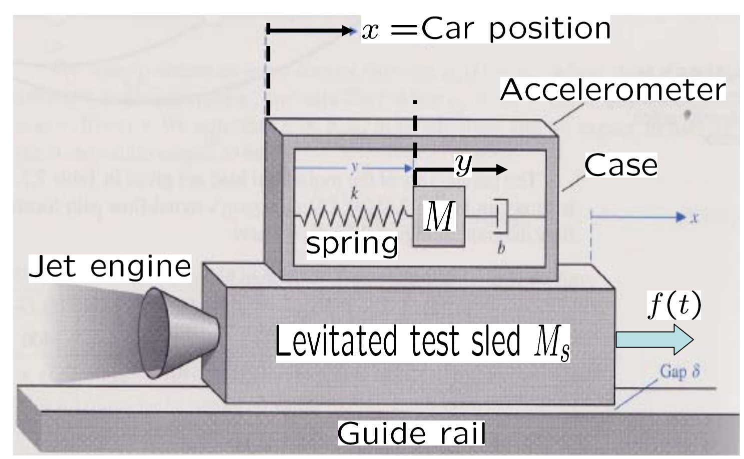
$$\Rightarrow \quad X(s) = \frac{1}{Ms^2}F(s) + \frac{x(0)}{s} + \frac{x'(0)}{s^2}$$

**(Total response)** = **(Forced response)** + **(Initial condition response)**

$$\Rightarrow \quad x(t) = \mathcal{L}^{-1}\left[\frac{1}{Ms^2}F(s)\right] + x(0)u_s(t) + x'(0)tu_s(t)$$



# Ex: Mechanical accelerometer



## Ex: Mechanical accelerometer (cont'd)

- We would like to know how y(t) moves when unit step f(t) is applied with zero ICs.
- By Newton's law

$$\begin{cases} M\dfrac{d^2}{dt^2}(x(t)+y(t)) = -b\dfrac{dy(t)}{dt} - ky(t) \\ M_s\dfrac{d^2x(t)}{dt^2} = f(t) \end{cases}$$

⇒

$\xrightarrow{\mathcal{L}}$ $Y(s) =$



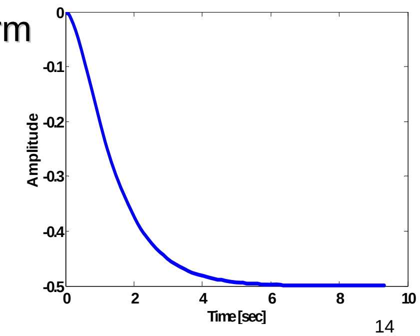
## Ex: Mechanical accelerometer (cont'd)

- Suppose that b/M=3, k/M=2 and Ms=1.
- Partial fraction expansion

$$Y(s) = -\frac{1}{s^2+3s+2}\cdot\frac{1}{s} =$$

- Inverse Laplace transform

$$y(t) =$$






## Summary & Exercises

- Solution procedure to ODEs
  1. Laplace transform
  2. Partial fraction expansion
  3. Inverse Laplace transform
- Next, modeling of physical systems using Laplace transform
- Exercises
  - Derive the solution to the accelerometer problem.
  - E2.4 in the textbook.