ME451: Control Systems

Lecture 15
Time response of 2nd-order systems

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Course roadmap

Modeling
- ✓ Laplace transform
- ✓ Transfer function
- ✓ Models for systems
  - ✓ electrical
  - ✓ mechanical
  - ✓ electromechanical
- ✓ Block diagrams
- ✓ Linearization

Analysis
- ✓ Time response
  - ➤ Transient
  - ✓ Steady state
- Frequency response
  - Bode plot
- ✓ Stability
  - ✓ Routh-Hurwitz
  - Nyquist

Design
- Design specs
- Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

(Matlab simulations &) laboratories

Performance measures (review)

- Transient response ← **(Today's lecture)**
  - Peak value
  - Peak time
  - Percent overshoot
  - Delay time
  - Rise time
  - Settling time
- Steady state response
  - Steady state error ← **(Done)**

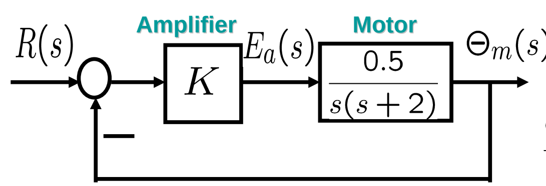
Next, we will connect these measures with s-domain.

Second-order systems

- A standard form of the second-order system

$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\qquad \begin{cases}\zeta: & \text{damping ratio}\\ \omega_n: & \text{undamped natural frequency}\end{cases}$$

- DC motor position control example

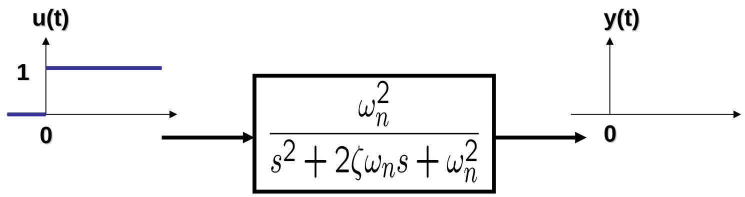


Closed-loop TF

$$\frac{\Theta_m(s)}{R(s)}=\frac{0.5K}{s^2+2s+0.5K}$$

Step response for 2nd-order system

- Input a unit step function to a 2nd-order system. What is the output?



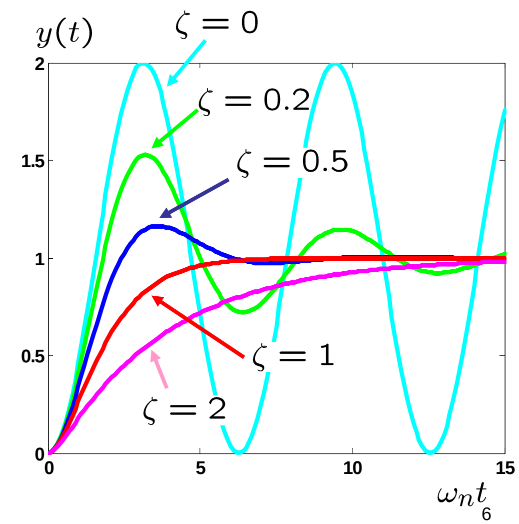
$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

DC gain

$$G(0) = 1 \quad \Rightarrow \quad \lim_{t\to\infty} y(t) = G(0) = 1 \text{ if } G \text{ is stable}$$

Step response for 2nd-order system for various damping ratio

- Undamped
  $\zeta = 0$
- Underdamped
  $0 < \zeta < 1$
- Critically damped
  $\zeta = 1$
- Overdamped
  $\zeta > 1$



Step response for 2nd-order system Underdamped case

- Math expression of y(t) for underdamped case

$$0 < \zeta < 1$$

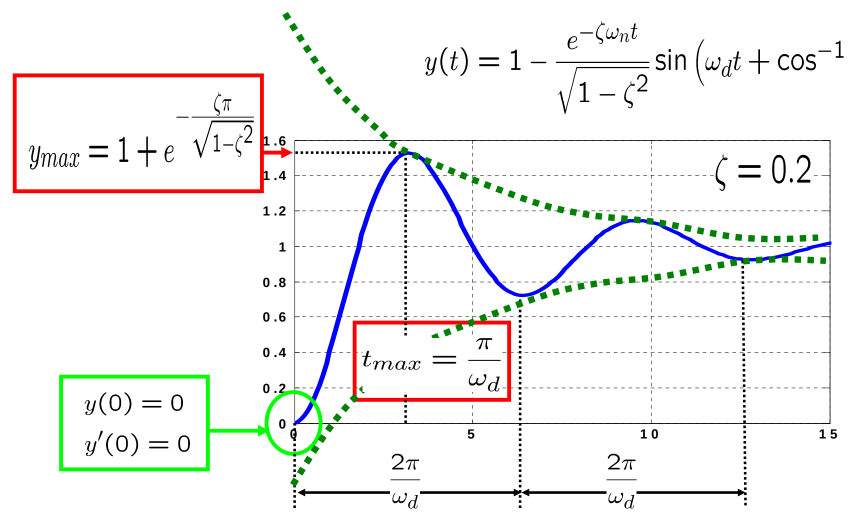
$$Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{1}{s}$$

$$\mathcal{L}^{-1} \Rightarrow \quad y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \cos^{-1}\zeta\right)$$

Damped natural frequency → $\omega_d := \omega_n\sqrt{1-\zeta^2}$

Peak value/time: Underdamped case

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \cos^{-1}\zeta\right)$$



Properties of 2nd-order system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2} \quad 0 < \zeta < 1$$

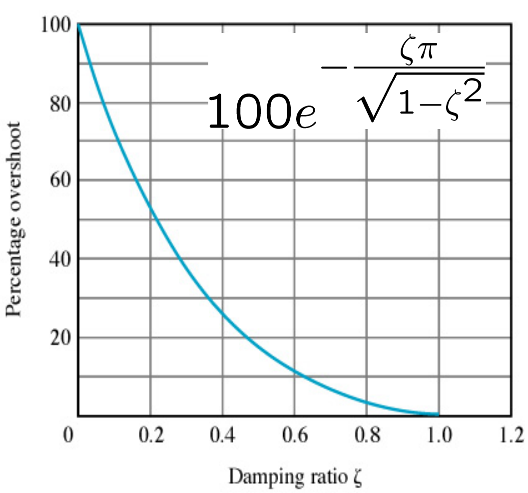
Peak time	$\frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$
Peak value	$1 + e^{-\zeta\pi/\sqrt{1-\zeta^2}}$
Percent overshoot	$100e^{-\zeta\pi/\sqrt{1-\zeta^2}}$
Settling time	$\approx \frac{3}{\zeta\omega_n}$ or $\frac{4}{\zeta\omega_n}$

(5%) **(2%)**

Some remarks

- Percent overshoot depends on ζ, but NOT ω_n.
- From 2nd-order transfer function, analytic expressions of delay & rise time are hard to obtain.
- Time constant is $1/(\zeta\omega_n)$, indicating convergence speed.
- For $\zeta>1$, we cannot define peak time, peak value, percent overshoot.

P.O. vs. damping ratio



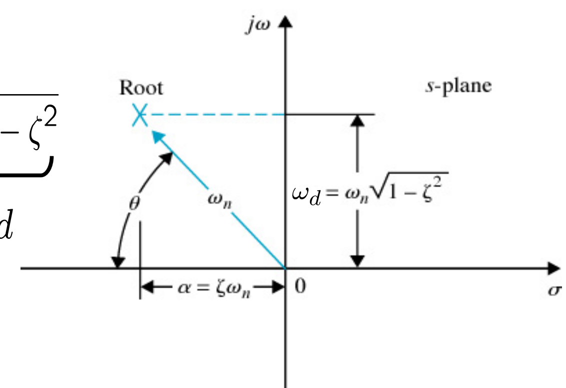
Pole locations of G

- Poles ($0<\zeta<1$)

$$s = -\zeta\omega_n \pm j\underbrace{\omega_n\sqrt{1-\zeta^2}}_{\omega_d}$$

- Damping ratio

$$\zeta = \cos\theta$$

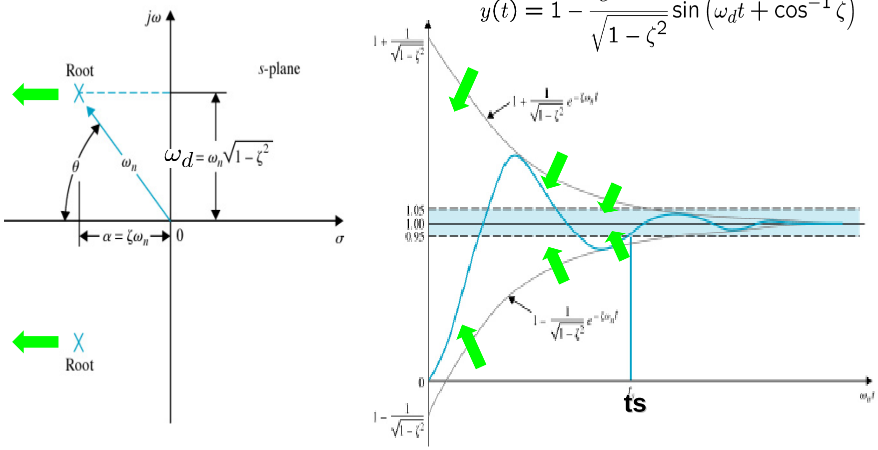


Next, we clarify the influence of pole location on step response.

Influence of real part of poles

- Settling time ts decreases.

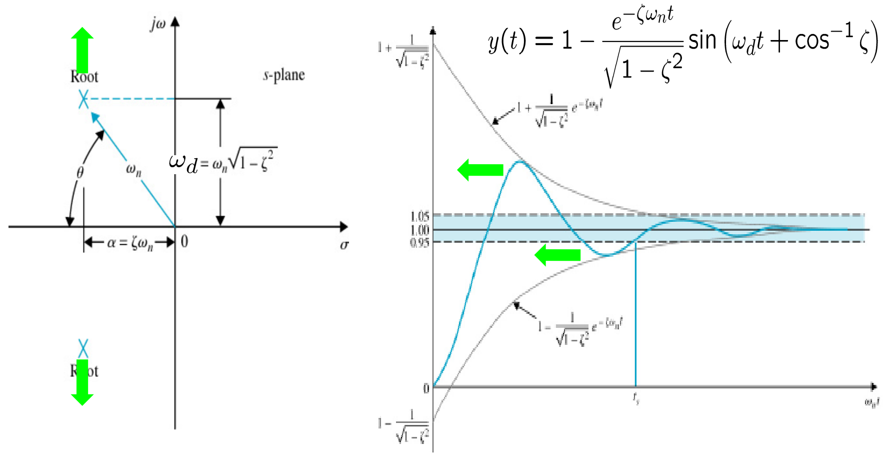
$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \cos^{-1}\zeta\right)$$



Influence of imag. part of poles

- Oscillation frequency ω_d increases.

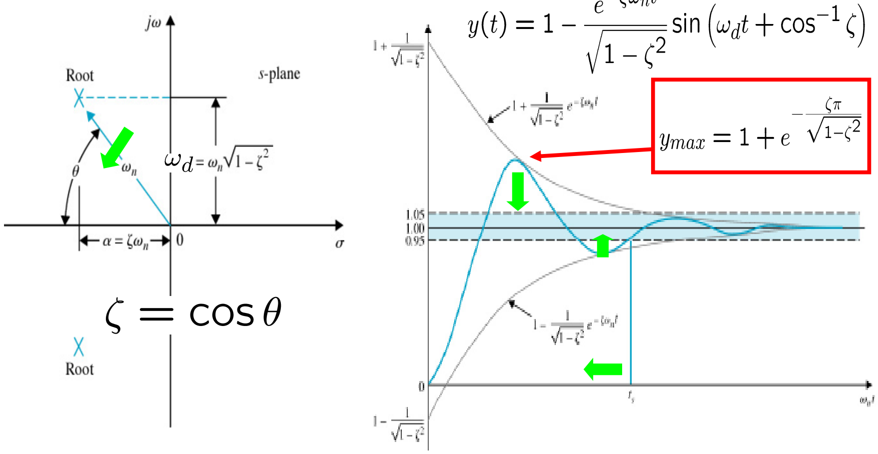
$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \cos^{-1}\zeta\right)$$



Influence of angle of poles

- Over/under-shoot decreases.

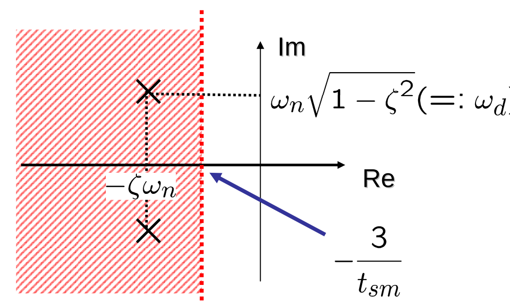
$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \cos^{-1}\zeta\right)$$



An example

- Require 5% settling time ts < tsm (given):

$$t_s \approx \frac{3}{\zeta\omega_n} < t_{sm} \quad \Rightarrow \quad \zeta\omega_n > \frac{3}{t_{sm}}$$

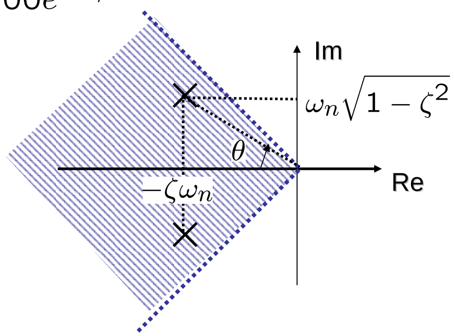


An example (cont'd)

- Require PO < POm (given):

$$PO = 100e^{-\zeta\pi/\sqrt{1-\zeta^2}} < PO_m$$
$$= 100e^{-\pi/\tan\theta}$$

→ $\theta < \theta_m$

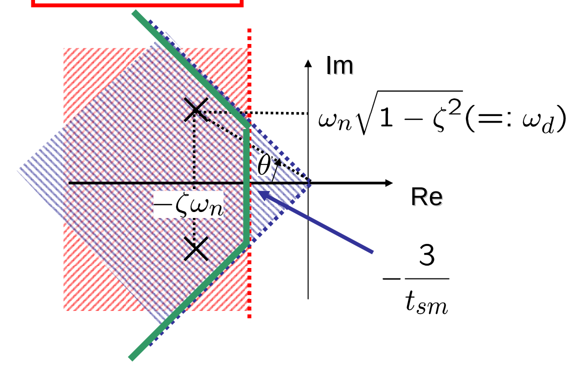


PO_m	θ_m
4.3%	$\pi/4$
16.3%	$\pi/3$

An example (cont'd)

- Combination of two requirements

$\zeta\omega_n > \frac{3}{t_{sm}}$ & $\theta < \theta_m$



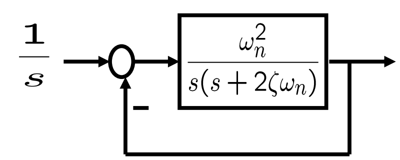
Summary

- Transient response of 2nd-order system is characterized by
  - Damping ratio ζ & undamped natural frequency ω_n
  - Pole locations
- Delay time and rise time are not so easy to characterize, and thus not covered in this course.
- For transient responses of high order systems, we need computer simulations.
- Next, Root locus

Exercises (Use a calculator if necessary.)

- Read the related topics from the textbook.

1. For the system below with ζ=0.6, ω_n=5 (rad/sec), obtain
   - Percent overshoot ?
   - 5% settling time ?



Exercises

2. For the system below, design K_1 and K_2 s.t.
   - Percent overshoot is at most 20%?
   - Peak time is at most 1 sec.?
   - With designed K_1 and K_2, what is 5% settling time?