ME451: Control Systems

Lecture 13
Steady-state error

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Course roadmap

Modeling
- Laplace transform
- Transfer function
- Models for systems
  - electrical
  - mechanical
  - electromechanical
- Block diagrams
- Linearization

Analysis
- Time response
  - Transient
  - Steady state
- Frequency response
  - Bode plot
- Stability
  - Routh-Hurwitz
  - Nyquist

Design
- Design specs
- Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

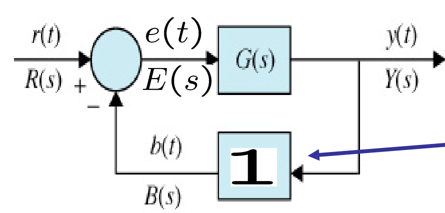
(Matlab simulations &) laboratories

Performance measures (review)

- Transient response ← **(From next lecture)**
  - Peak value
  - Peak time
  - Percent overshoot
  - Delay time
  - Rise time
  - Settling time
- Steady state response
  - Steady state error ← **(Today's lecture)**

Next, we will connect these measures with s-domain.

Steady-state error: *unity feedback*



We assume that the CL system is stable!

Unity feedback!

- Suppose that we want output y(t) to track r(t).
- Error $e(t) = r(t) - y(t)$
- Steady-state error

$$e_{ss} := \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{1}{1+G(s)}R(s)$$

Final value theorem
(Suppose CL system is stable!!!)

Error constants

- Step-error (position-error) constant

$$K_p := \lim_{s \to 0} G(s)$$

- Ramp-error (velocity-error) constant

$$K_v := \lim_{s \to 0} sG(s)$$

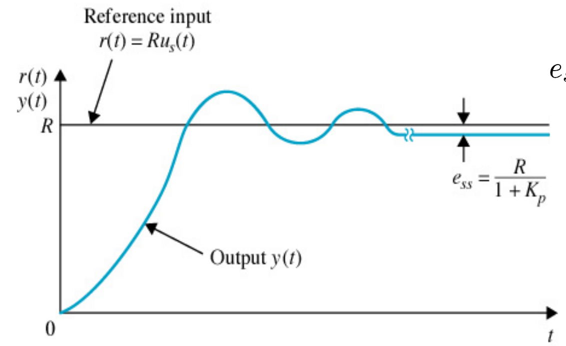
- Parabolic-error (acceleration-error) constant

$$K_a := \lim_{s \to 0} s^2 G(s)$$

- Kp, Kv, Ka : *ability to reduce steady-state error*

Steady-state error for step r(t)

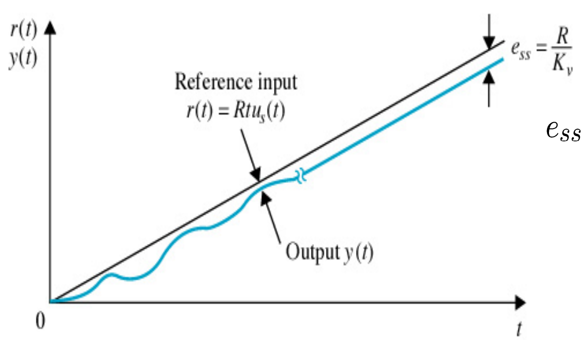
$$\boxed{r(t) = Ru_s(t) \Rightarrow e_{ss} = \frac{R}{1+K_p}}$$



$$\begin{aligned} e_{ss} &= \lim_{s \to 0} s \frac{1}{1+G(s)} \cdot \frac{R}{s} \\ &= \frac{R}{1+\underbrace{G(0)}_{Kp}} \end{aligned}$$

Steady-state error for ramp r(t)

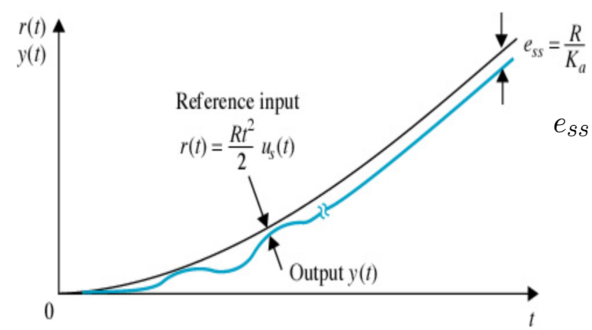
$$\boxed{r(t) = Rtu_s(t) \Rightarrow e_{ss} = \frac{R}{K_v}}$$



$$\begin{aligned} e_{ss} &= \lim_{s \to 0} s \frac{1}{1+G(s)} \cdot \frac{R}{s^2} \\ &= \frac{R}{\underbrace{\lim_{s \to 0} sG(s)}_{Kv}} \end{aligned}$$

Steady-state error for parabolic r(t)

$$\boxed{r(t) = \frac{Rt^2}{2} u_s(t) \Rightarrow e_{ss} = \frac{R}{K_a}}$$



$$\begin{aligned} e_{ss} &= \lim_{s \to 0} s \frac{1}{1+G(s)} \cdot \frac{R}{s^3} \\ &= \frac{R}{\underbrace{\lim_{s \to 0} s^2 G(s)}_{Ka}} \end{aligned}$$

System type

- ***System type of G*** is defined as the order (number) of poles of G(s) at s=0.
- Examples

$$G(s)=\frac{K(1+0.5s)}{s(1+s)(1+2s)(1+s+s^2)} \quad \Rightarrow \quad \text{type 1}$$

$$G(s)=\frac{K(1+s)}{s^2}e^{-Ts} \quad \Rightarrow \quad \text{type 2}$$

$$G(s)=\frac{K(1+2s)}{s^3} \quad \Rightarrow \quad \text{type 3}$$

Zero steady-state error

- If error constant is infinite, we can achieve zero steady-state error. (Accurate tracking)
  - For step r(t)

$$K_p=\lim_{s\to 0}G(s)=\infty \Leftrightarrow G(s) \text{ is of at least type 1}$$

  - For ramp r(t)

$$K_v=\lim_{s\to 0}sG(s)=\infty \Leftrightarrow G(s) \text{ is of at least type 2}$$

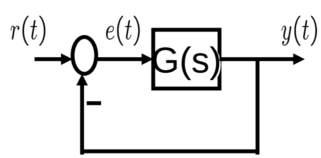
  - For parabolic r(t)

$$K_a=\lim_{s\to 0}s^2G(s)=\infty \Leftrightarrow G(s) \text{ is of at least type 3}$$

Example 1

- G(s) of type 2

$$G(s)=\frac{K}{s^2(s+12)}$$



- Characteristic equation

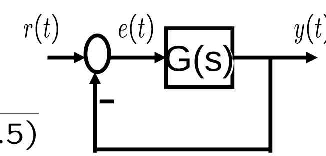
$$1+G(s)=0 \Leftrightarrow s^2(s+12)+K=0 \Leftrightarrow s^3+12s^2+K=0$$

- CL system is NOT stable for any K.
- e(t) goes to infinity. (Don't use today's results if CL system is not stable!!!)

Example 2

- G(s) of type 1

$$G(s)=\frac{K(s+3.15)}{s(s+1.5)(s+0.5)}$$



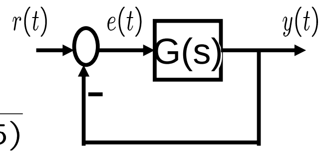
- By Routh-Hurwitz criterion, CL is stable iff

$$0<K<1.304$$

- Step r(t) $\quad e_{ss}=\frac{R}{1+K_p}=0$
- Ramp r(t) $\quad e_{ss}=\frac{R}{K_v} \qquad K_v:=\lim_{s\to 0}sG(s)=\frac{3.15K}{0.75}=4.2K$
- Parabolic r(t) $\quad e_{ss}=\frac{R}{K_a}=\infty \qquad K_a:=\lim_{s\to 0}s^2G(s)=0$

Example 3

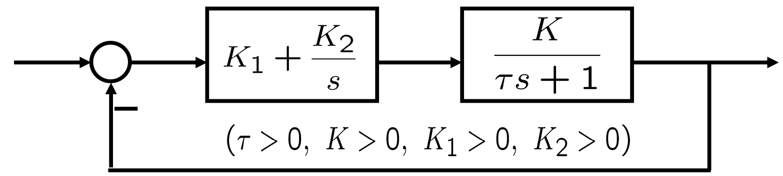
- G(s) of type 2



$$G(s) = \frac{5(s+1)}{s^2(s+12)(s+5)}$$

- By Routh-Hurwitz criterion, we can show that CL system is stable.
- Step r(t) $\quad e_{ss} = \frac{R}{1+K_p} = 0$
- Ramp r(t) $\quad e_{ss} = \frac{R}{K_v} = 0$
- Parabolic r(t) $\quad e_{ss} = \frac{R}{K_a} = 12R \qquad K_a := \lim_{s \to 0} s^2 G(s) = \frac{1}{12}$

A control example



- Closed-loop stable?
- Compute error constants

$K_p = \qquad\qquad K_v = \qquad\qquad K_a =$

- Compute steady state errors

$e_{ss} = \qquad\qquad e_{ss} = \qquad\qquad e_{ss} =$

Summary and Exercises

- Steady-state error
  - For unity feedback (STABLE!) systems, the system type of the forward-path system determines if the steady-state error is zero.
  - The key tool is the final value theorem!
- Next, time response of 1st-order systems
- Exercises
  - Go over the examples in this lecture.