ME451: Control Systems

Lecture 11
Routh-Hurwitz criterion: Control examples

Dr. Jongeun Choi
Department of Mechanical Engineering
Michigan State University

Course roadmap

Modeling
- ✔ Laplace transform
- ✔ Transfer function
- ✔ Models for systems
  - ✔ electrical
  - ✔ mechanical
  - ✔ electromechanical
- ✔ Block diagrams
- ✔ Linearization

Analysis
- Time response
  - • Transient
  - • Steady state
- Frequency response
  - • Bode plot
- ✔ Stability
  - • Routh-Hurwitz
  - • Nyquist

Design
- Design specs
- Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

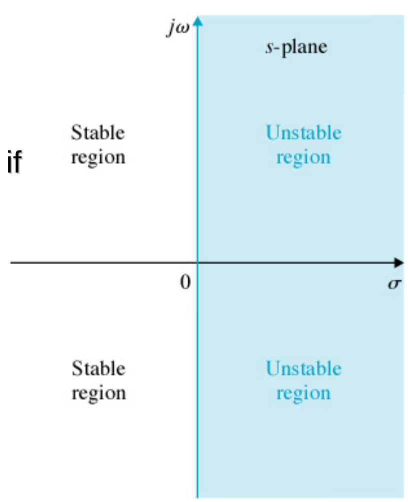
(Matlab simulations &) laboratories

Stability summary (review)

Let s_i be poles of rational G. Then, G is …

- (BIBO, asymptotically) stable if Re(s_i)<0 for all i.
- marginally stable if
  - Re(s_i)<=0 for all i, and
  - simple root for Re(s_i)=0
- unstable if

  it is neither stable nor marginally stable.



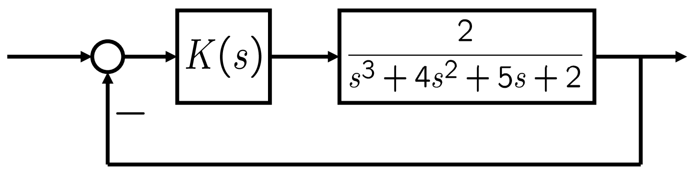
Routh-Hurwitz criterion (review)

$$Q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	$\cdots$
s^{n-2}	b_1	b_2	b_3	b_4	$\cdots$
s^{n-3}	c_1	c_2	c_3	c_4	$\cdots$
$\vdots$	$\vdots$	$\vdots$			
s^2	k_1	k_2			
s^1	l_1				
s^0	m_1				

The number of roots in the right half-plane is equal to the number of sign changes in the first column of Routh array.

Example 1



- Design K(s) that stabilizes the closed-loop system for the following cases.
  - K(s) = K (constant)
  - K(s) = $K_P + K_I/s$ (PI (Proportional-Integral) controller)

Example 1: K(s)=K

- Characteristic equation

$$1 + K\frac{2}{s^3+4s^2+5s+2} = 0$$

$$\Rightarrow \quad s^3+4s^2+5s+2+2K=0$$

- Routh array

$$\begin{array}{c|ll} s^3 & 1 & 5 \\ s^2 & 4 & 2+2K \\ s^1 & \frac{18-2K}{4} & \\ s^0 & 2+2K & \end{array} \quad \Rightarrow \quad -1 < K < 9$$

Example 1: K(s)=KP+KI/s

- Characteristic equation

$$1 + \left(K_P + \frac{K_I}{s}\right)\frac{2}{s^3+4s^2+5s+2} = 0$$

$$\Rightarrow \quad s^4+4s^3+5s^2+(2+2K_P)s+2K_I=0$$

- Routh array

$$\begin{array}{c|lll} s^4 & 1 & 5 & 2K_I \\ s^3 & 4 & 2+2K_P & \\ s^2 & \frac{18-2K_P}{4} & 2K_I & \\ s^1 & * & & \\ s^0 & 2K_I & & \end{array}$$

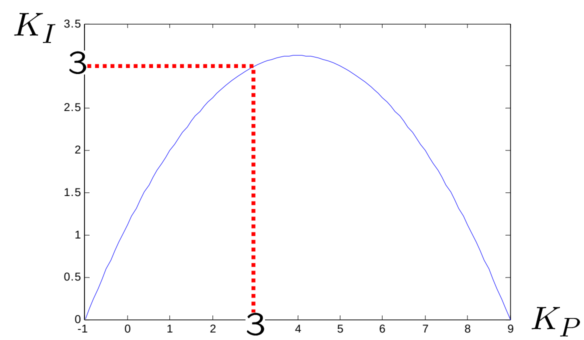
$\rightarrow K_P < 9$

$\rightarrow K_I > 0$

Example 1: Range of (KP,KI)

- From Routh array,

$$K_P < 9$$
$$K_I > 0$$
$$(1+K_P)(9-K_P)-8K_I > 0$$



Example 1: K(s)=KP+KI/s (cont'd)

- Select KP=3 (<9)
- Routh array (cont'd)

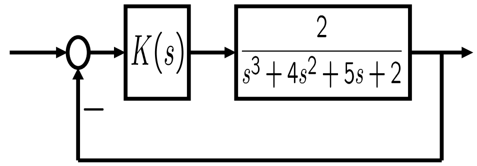
s^4	1	5	$2K_I$
s^3	4	8	
s^2	3	$2K_I$	
s^1	$\frac{24-8K_I}{3}$		
s^0	$2K_I$		

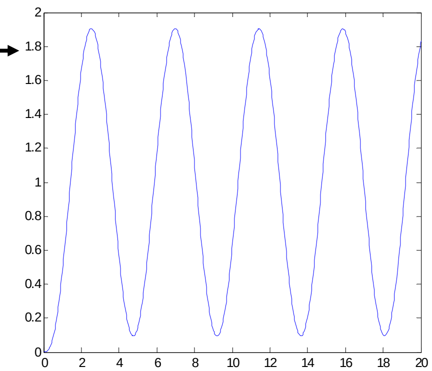
s^1, s^0 rows $\rightarrow$ $0 < K_I < 3$

- If we select different KP, the range of KI changes.

Example 1: What happens if KP=KI=3

- Auxiliary equation $3s^2 + 6 = 0 \Leftrightarrow s = \pm\sqrt{2}j$



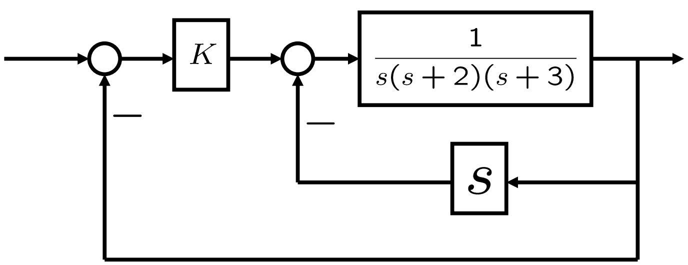


Unit step response

- Oscillation frequency

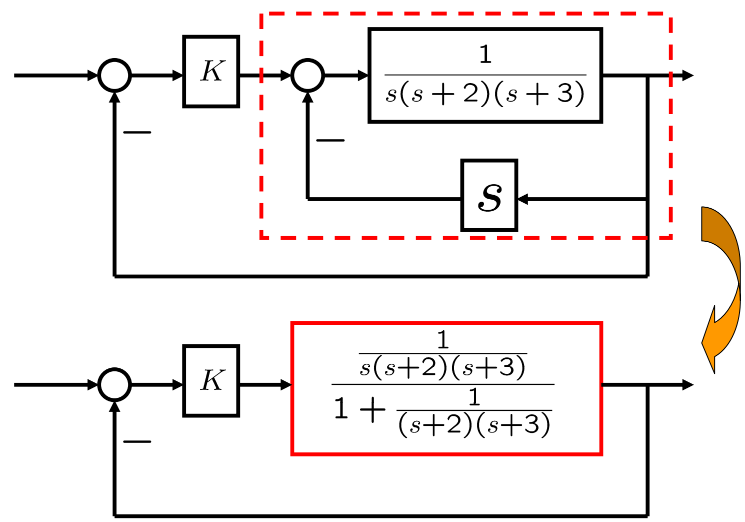
  $\sqrt{2}(\text{rad/sec})$
- Period $\frac{2\pi}{\sqrt{2}} \approx 4.4(\text{sec})$

Example 2



- Determine the range of K and a that stabilize the closed-loop system.

Example 2 (cont'd)



Example 2 (cont'd)

- Characteristic equation

$$1+K\frac{\frac{1}{s(s+2)(s+3)}}{1+\frac{1}{(s+2)(s+3)}}=0$$

$$\Rightarrow\ 1+\frac{K}{s}\cdot\frac{1}{(s+2)(s+3)+1}=0$$

$$\Rightarrow\ s(s+2)(s+3)+s+K=0$$

$$\Rightarrow\ s^3+5s^2+7s+K=0$$

Example 2 (cont'd)

- Routh array $s^3+5s^2+7s+K=0$

s^3	1	7
s^2	5	K
s^1	$\frac{35-K}{5}$	
s^0	K	

$\longrightarrow\ 0<K<35$

- If K=35, oscillation frequency is obtained by the auxiliary equation

$$5s^2+35=0 \Leftrightarrow s=\pm\sqrt{7}j$$

Summary and Exercises

- Control examples for Routh-Hurwitz criterion
  - P controller gain range for stability
  - PI controller gain range for stability
  - Oscillation frequency
  - Characteristic equation
- Next
  - Time domain specifications
- Exercises

More example 1

$$Q(s)=s^3+s^2+s+1\ \ (=(s+1)(s^2+1))$$

Routh array

s^3	1	1
s^2	1	1
s^1	~~0~~ **2**	
s^0	1	

Derivative of auxiliary poly.
$(s^2+1)'=2s$

(Auxiliary poly. is a factor of Q(s).)

No sign changes in the first column $\Rightarrow$ No root in OPEN(!) RHP

More example 2

$$Q(s) = s^5+s^4+2s^3+2s^2+s+1 \ (= (s+1)(s^2+1)^2)$$

Routh array

s^5	1	2	1
s^4	1	2	1
s^3	~~0~~ **4**	~~0~~ **4**	
s^2	1	1	
s^1	~~0~~ **2**		
s^0	1		

Derivative of auxiliary poly.

$$(s^4+2s^2+1)' = 4s^3+4s$$

$$(s^2+1)' = 2s$$

No sign changes
in the first column

No root in OPEN(!) RHP

More example 3

$$Q(s) = s^4 - 1 \ (= (s+1)(s-1)(s^2+1))$$

Routh array

s^4	1	0	-1
s^3	~~0~~ **4**	~~0~~ **0**	
s^2	~~0~~ ε	-1	
s^1	$4/\varepsilon$		
s^0	-1		

Derivative of auxiliary poly.

$$(s^4-1)' = 4s^3$$

One sign changes
in the first column

One root in OPEN(!) RHP