ME451: Control Systems

Lecture 10
Routh-Hurwitz stability criterion

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Course roadmap

Modeling	*Analysis*	*Design*
✔ Laplace transform	Time response • Transient • Steady state	Design specs
✔ Transfer function	Frequency response • Bode plot	Root locus
✔ Models for systems ✔ electrical ✔ mechanical ✔ electromechanical	✔ Stability • Routh-Hurwitz • Nyquist	Frequency domain PID & Lead-lag Design examples
✔ Linearization		

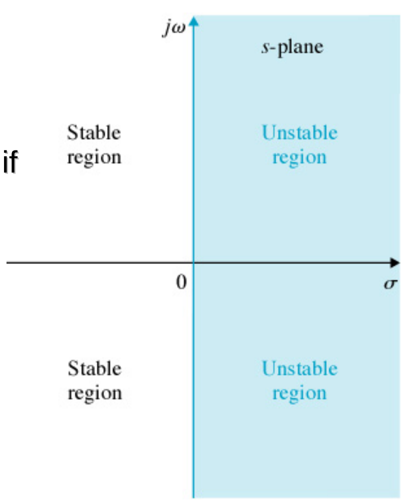
(Matlab simulations &) laboratories

Stability summary (review)

Let s_i be poles of rational G. Then, G is …

- (BIBO, asymptotically) stable if $Re(s_i)<0$ for all i.
- marginally stable if
  - $Re(s_i)<=0$ for all i, and
  - simple root for $Re(s_i)=0$
- unstable if

  it is neither stable nor marginally stable.



Routh-Hurwitz criterion

- This is for LTI systems with a *polynomial* denominator (without sin, cos, exponential etc.)
- It determines if all the roots of a polynomial
  - lie in the open LHP (left half-plane),
  - or equivalently, have negative real parts.
- It also determines the number of roots of a polynomial in the open RHP (right half-plane).
- It does NOT explicitly compute the roots.

Polynomial and an assumption

- Consider a polynomial

$$Q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

- Assume $a_0 \neq 0$
  - If this assumption does not hold, Q can be factored as

$$Q(s) = s^m \underbrace{\left(\hat{a}_{n-m} s^{n-m} + \cdots + \hat{a}_1 s + \hat{a}_0\right)}_{\hat{Q}(s)}$$

  where $\hat{a}_0 \neq 0$
  - The following method applies to the polynomial $\hat{Q}(s)$

Routh array

$$\begin{array}{c|ccccc}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & \vdots & \vdots & & & \\
s^2 & k_1 & k_2 & & & \\
s^1 & l_1 & & & & \\
s^0 & m_1 & & & &
\end{array}$$

From the given polynomial

Routh array (How to compute the third row)

$$\begin{array}{c|ccccc}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & \vdots & \vdots & & & \\
s^2 & k_1 & k_2 & & & \\
s^1 & l_1 & & & & \\
s^0 & m_1 & & & &
\end{array}$$

$$\begin{aligned}
b_1 &= \frac{a_{n-2}a_{n-1} - a_n a_{n-3}}{a_{n-1}} \\
b_2 &= \frac{a_{n-4}a_{n-1} - a_n a_{n-5}}{a_{n-1}} \\
&\vdots
\end{aligned}$$

Routh array (How to compute the fourth row)

$$\begin{array}{c|ccccc}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & \vdots & \vdots & & & \\
s^2 & k_1 & k_2 & & & \\
s^1 & l_1 & & & & \\
s^0 & m_1 & & & &
\end{array}$$

$$\begin{aligned}
c_1 &= \frac{a_{n-3}b_1 - a_{n-1}b_2}{b_1} \\
c_2 &= \frac{a_{n-5}b_1 - a_{n-1}b_3}{b_1} \\
&\vdots
\end{aligned}$$

Routh-Hurwitz criterion

$$
\begin{array}{l|lllll}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & \vdots & \vdots & & & \\
s^2 & k_1 & k_2 & & & \\
s^1 & l_1 & & & & \\
s^0 & m_1 & & & &
\end{array}
$$

The number of roots in the open right half-plane is equal to the number of sign changes in the first column of Routh array.

Example 1

$$Q(s) = s^3 + s^2 + 2s + 8 \;(= (s+2)(s^2 - s + 4))$$

Routh array

$$
\begin{array}{l|ll}
s^3 & 1 & 2 \\
s^2 & 1 & 8 \\
s^1 & -6 & \\
s^0 & 8 &
\end{array}
$$

$$\frac{2-8}{1} \qquad \frac{8 \times (-6) - 0}{-6}$$

Two sign changes in the first column $1 \to -6 \to 8$ ➡ Two roots in RHP $\frac{1}{2} \pm \frac{j\sqrt{15}}{2}$

Example 2

$$Q(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$$

Routh array

$$
\begin{array}{l|lll}
s^5 & 1 & 2 & 11 \\
s^4 & 2 & 4 & 10 \\
s^3 & \cancel{0}\,\varepsilon & 6 & \\
s^2 & \frac{4\varepsilon-12}{\varepsilon} & 10 & \\
s^1 & 6 & & \\
s^0 & 10 & &
\end{array}
$$

If 0 appears in the first column of a nonzero row in Routh array, replace it with a small positive number. In this case, Q has some roots in RHP.

Two sign changes in the first column ➡ Two roots in RHP

$$\varepsilon \to \underbrace{\frac{4\varepsilon - 12}{\varepsilon}}_{<0} \to 6$$

Example 3

$$Q(s) = s^4 + s^3 + 3s^2 + 2s + 2$$

Routh array

$$
\begin{array}{l|lll}
s^4 & 1 & 3 & 2 \\
s^3 & 1 & 2 & \\
s^2 & 1 & 2 & \\
s^1 & \cancel{0}\;2 & & \\
s^0 & 2 & &
\end{array}
$$

If zero row appears in Routh array, Q has roots either on the imaginary axis or in RHP.

No sign changes in the first column ➡ No roots in RHP

But some roots are on imag. axis.

Take derivative of an *auxiliary polynomial* (which is a factor of Q(s)) $s^2 + 2$

Example 4

$$Q(s) = s^3 + 3Ks^2 + (K+2)s + 4$$

Find the range of K s.t. Q(s) has all roots in the left half plane. (Here, K is a design parameter.)

Routh array

s^3	1	$K+2$
s^2	$3K$	4
s^1	$\frac{3K(K+2)-4}{3K}$	
s^0	4	

No sign changes in the first column

$$\Rightarrow \begin{cases} 3K > 0 \\ 3K(K+2) - 4 > 0 \end{cases}$$

$$\Rightarrow K > -1 + \frac{\sqrt{21}}{3}$$

Simple & important criteria for stability

- 1st order polynomial $Q(s) = a_1 s + a_0$

  All roots are in LHP $\Leftrightarrow$ a_1 and a_0 have the same sign

- 2nd order polynomial $Q(s) = a_2 s^2 + a_1 s + a_0$

  All roots are in LHP $\Leftrightarrow$ a_2, a_1 and a_0 have the same sign

- Higher order polynomial $Q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$

  All roots are in LHP $\Rightarrow$ All a_k have the same sign

Examples

$Q(s)$	All roots in open LHP?
$3s + 5$	**Yes / No**
$-2s^2 - 5s - 100$	**Yes / No**
$523s^2 - 57s + 189$	**Yes / No**
$(s^2 + s - 1)(s^2 + s + 1)$	**Yes / No**
$s^3 + 5s^2 + 10s - 3$	**Yes / No**

Summary and Exercises

- Routh-Hurwitz stability criterion
  - Routh array
  - Routh-Hurwitz criterion is applicable to only polynomials (so, it is not possible to deal with exponential, sin, cos etc.).
- Next,
  - Routh-Hurwitz criterion in control examples
- Exercises
  - Read Routh-Hurwitz criterion in the textbook.
  - Do Examples.