

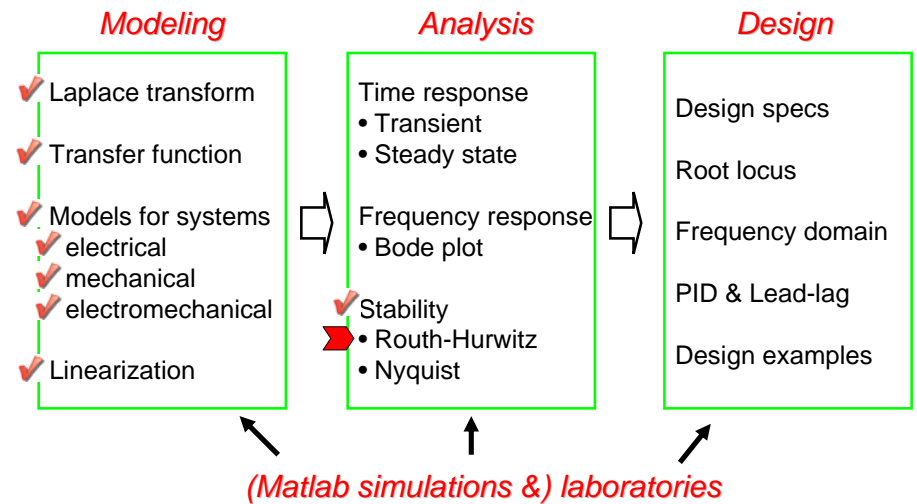
ME451: Control Systems

Lecture 10

Routh-Hurwitz stability criterion

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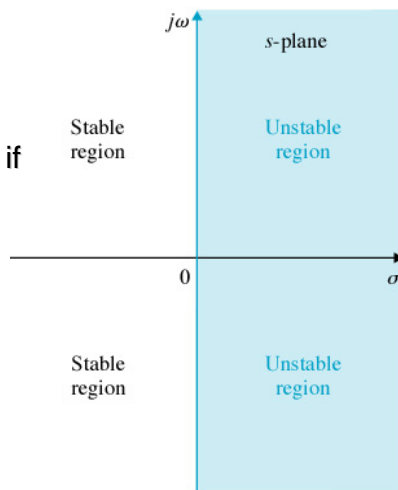
Course roadmap



Stability summary (review)

Let s_i be **poles** of rational G . Then, G is ...

- **(BIBO, asymptotically) stable** if $\text{Re}(s_i) < 0$ for all i .
- **marginally stable** if
 - $\text{Re}(s_i) \leq 0$ for all i , and
 - simple root for $\text{Re}(s_i) = 0$
- **unstable** if it is neither stable nor marginally stable.



Routh-Hurwitz criterion

- This is for LTI systems with a *polynomial* denominator (without sin, cos, exponential etc.)
- It determines if all the roots of a polynomial
 - lie in the open LHP (left half-plane),
 - or equivalently, have negative real parts.
- It also determines the number of roots of a polynomial in the open RHP (right half-plane).
- It does **NOT** explicitly compute the roots.

Polynomial and an assumption

- Consider a polynomial

$$Q(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0$$

- Assume $a_0 \neq 0$

- If this assumption does not hold, Q can be factored as

$$Q(s) = s^m \underbrace{(\hat{a}_{n-m} s^{n-m} + \dots + \hat{a}_1 s + \hat{a}_0)}_{\hat{Q}(s)}$$

where $\hat{a}_0 \neq 0$

- The following method applies to the polynomial $\hat{Q}(s)$

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Routh array

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	\dots	From the given polynomial
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	\dots	
s^{n-2}	b_1	b_2	b_3	b_4	\dots	
s^{n-3}	c_1	c_2	c_3	c_4	\dots	
\vdots	\vdots	\vdots				
s^2	k_1	k_2				
s^1	l_1					
s^0	m_1					

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Routh array (How to compute the third row)

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	\dots	}
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	\dots	
s^{n-2}	b_1	b_2	b_3	b_4	\dots	}
s^{n-3}	c_1	c_2	c_3	c_4	\dots	
\vdots	\vdots	\vdots				
s^2	k_1	k_2				
s^1	l_1					
s^0	m_1					

$$b_1 = \frac{a_{n-2}a_{n-1} - a_n a_{n-3}}{a_{n-1}}$$

$$b_2 = \frac{a_{n-4}a_{n-1} - a_n a_{n-5}}{a_{n-1}}$$

$$\vdots$$

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Routh array (How to compute the fourth row)

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	\dots	}
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	\dots	
s^{n-2}	b_1	b_2	b_3	b_4	\dots	}
s^{n-3}	c_1	c_2	c_3	c_4	\dots	
\vdots	\vdots	\vdots				
s^2	k_1	k_2				
s^1	l_1					
s^0	m_1					

$$c_1 = \frac{a_{n-3}b_1 - a_{n-1}b_2}{b_1}$$

$$c_2 = \frac{a_{n-5}b_1 - a_{n-1}b_3}{b_1}$$

$$\vdots$$

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Routh-Hurwitz criterion

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	\dots
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	\dots
s^{n-2}	b_1	b_2	b_3	b_4	\dots
s^{n-3}	c_1	c_2	c_3	c_4	\dots
\vdots	\vdots	\vdots			
s^2	k_1	k_2			
s^1	l_1				
s^0	m_1				

The number of roots in the open right half-plane is equal to the number of sign changes in the **first column** of Routh array.

Example 1

$$Q(s) = s^3 + s^2 + 2s + 8 (= (s + 2)(s^2 - s + 4))$$

Routh array

s^3	
s^2	
s^1	
s^0	

Two sign changes in the first column
 $1 \rightarrow -6 \rightarrow 8$



Two roots in RHP
 $\frac{1}{2} \pm \frac{j\sqrt{15}}{2}$

Example 2

$$Q(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$$

Routh array

s^5	
s^4	
s^3	
s^2	
s^1	
s^0	

If 0 appears in the first column of a nonzero row in Routh array, replace it with a small positive number. In this case, Q has some roots in RHP.

Two sign changes in the first column \rightarrow Two roots in RHP

$$\epsilon \rightarrow \frac{4\epsilon - 12}{\underbrace{\epsilon}_{<0}} \rightarrow 6$$

Example 3

$$Q(s) = s^4 + s^3 + 3s^2 + 2s + 2$$

Routh array

s^4	
s^3	
s^2	
s^1	
s^0	

If zero row appears in Routh array, Q has roots either on the imaginary axis or in RHP.

No sign changes in the first column \rightarrow No roots in RHP

Take derivative of an auxiliary polynomial (which is a factor of Q(s)) $s^2 + 2$

But some roots are on imag. axis.

Example 4

$$Q(s) = s^3 + 3Ks^2 + (K + 2)s + 4$$

Find the range of K s.t. Q(s) has all roots in the left half plane. (Here, K is a design parameter.)

Routh array

s^3	
s^2	
s^1	
s^0	

No sign changes in the first column

$$\begin{cases} 3K > 0 \\ 3K(K + 2) - 4 > 0 \end{cases}$$

$$\Rightarrow K > -1 + \frac{\sqrt{21}}{3}$$

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Simple & important criteria for stability

- 1st order polynomial $Q(s) = a_1s + a_0$
All roots are in LHP $\Leftrightarrow a_1$ and a_0 have the same sign
- 2nd order polynomial $Q(s) = a_2s^2 + a_1s + a_0$
All roots are in LHP $\Leftrightarrow a_2, a_1$ and a_0 have the same sign
- Higher order polynomial $Q(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0$
All roots are in LHP \Leftrightarrow All a_k have the same sign

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Examples

$Q(s)$	All roots in open LHP?
$3s + 5$	Yes / No
$-2s^2 - 5s - 100$	Yes / No
$523s^2 - 57s + 189$	Yes / No
$(s^2 + s - 1)(s^2 + s + 1)$	Yes / No
$s^3 + 5s^2 + 10s - 3$	Yes / No

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Summary and Exercises

- Routh-Hurwitz stability criterion
 - Routh array
 - Routh-Hurwitz criterion is applicable to only polynomials (so, it is not possible to deal with exponential, sin, cos etc.).
- Next,
 - Routh-Hurwitz criterion in control examples
- Exercises
 - Read Routh-Hurwitz criterion in the textbook.
 - Do Examples.

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