Linearization

Reference: Sections 2.10-2.11, *Control Systems Engineering*, N.S. Nise

The analytical models for *most* mechanical engineering systems are non-linear. A linear system has output(s) linearly dependent on its input(s) through a proportionality constant. A linear system possesses two properties: *superposition* and *homogeneity*.

Superposition means

If input $r_1(t)$ yields output $c_1(t)$, and input $r_2(t)$ yields output $c_2(t)$	Then input $r_1(t)+r_2(t)$ yields output $c_1(t)+c_2(t)$

Homogeneity means

If input $r_1(t)$ yields output $c_1(t)$	Then For a constant A, input $Ar_1(t)$ yields output $Ac_1(t)$

The linear function in Figure 1obeys these properties while the not-linear (affine) and non-linear functions do not.

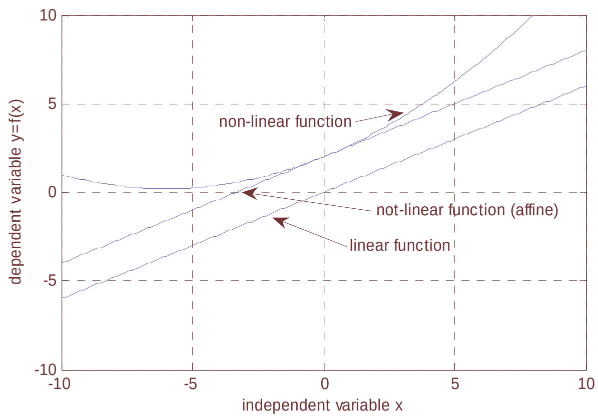


Figure 1: Linear, Affine and Non-linear Function Examples

To linearize functions, we use a Taylor expansion about an operating point. The Taylor expansion requires a new set of coordinates $(\tilde{x},\tilde{y})$ defined such that the linearized function passes through the origin of this new coordinate system.

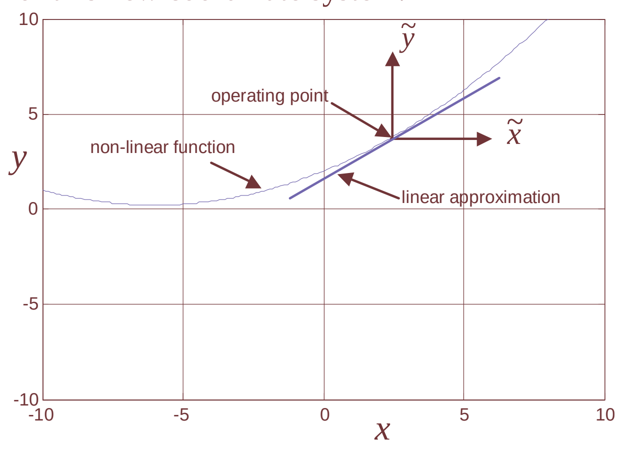


Figure 2: Operating Point Linear Approximation found through Taylor Expansion

The Six Steps of Linearization

1) Identify the system model's input $r(t)$ and output $c(t)$.

2) Express Model in the form $f(r,\dot{r},\ddot{r},\ldots,c,\dot{c},\ddot{c},\ldots)=0$.

3) Define an equilibrium operating point where all input and output derivatives are zero ($\dot{r}=\ddot{r}=\ldots=\dot{c}=\ddot{c}=\ldots=0$) and the operating point (r_o,c_o) satisfies the original model such that $f(r,\dot{r},\ddot{r},\ldots,c,\dot{c},\ddot{c},\ldots)=f(r_o,0,0,\ldots,c_o,0,0,\ldots)=0$ at the input/output values $(r,c)=(r_o,c_o)$.

4) Perform a Taylor Series expansion about the operating point (r_o,c_o) retaining only 1st derivative terms.

$$
\begin{aligned}
f(r,\dot{r},\ddot{r},\ldots,c,\dot{c},\ddot{c},\ldots) \cong\ & f(r_o,0,0,\ldots,c_o,0,0,\ldots) \\
& +\left.\frac{\partial f}{\partial r}\right|_{(r_o,c_o)}(r-r_o)+\left.\frac{\partial f}{\partial \dot{r}}\right|_{(r_o,c_o)}(\dot{r}-\dot{r}_o)+\left.\frac{\partial f}{\partial \dot{r}}\right|_{(r_o,c_o)}(\ddot{r}-\ddot{r}_o)+\ldots \\
& +\left.\frac{\partial f}{\partial c}\right|_{(r_o,c_o)}(c-c_o)+\left.\frac{\partial f}{\partial \dot{r}}\right|_{(r_o,c_o)}(\dot{c}-\dot{c}_o)+\left.\frac{\partial f}{\partial \dot{r}}\right|_{(r_o,c_o)}(\ddot{c}-\ddot{c}_o)+\ldots
\end{aligned}
$$

5) Change variables from original input $r(t)$ and output $c(t)$ to deviations about the defined operating point. These new variables are the differences required in the Taylor expansion.

$$\tilde{r}=(r-r_o),\ \dot{\tilde{r}}=(\dot{r}-\dot{r}_o),\ \ddot{\tilde{r}}=(\ddot{r}-\ddot{r}_o),\ etc.$$

$$\tilde{c}=(c-c_o),\ \dot{\tilde{c}}=(\dot{c}-\dot{c}_o),\ \ddot{\tilde{c}}=(\ddot{c}-\ddot{c}_o),\ etc.$$

with $f(r_o,0,0,\ldots,c_o,0,0,\ldots)=0$ from step 3 yields

$$
\begin{aligned}
f(\tilde{r},\dot{\tilde{r}},\ddot{\tilde{r}},\ldots,\tilde{c},\dot{\tilde{c}},\ddot{\tilde{c}},\ldots) \cong 0 & +\left[\left.\frac{\partial f}{\partial r}\right|_{(r_o,c_o)}\right]\tilde{r}+\left[\left.\frac{\partial f}{\partial \dot{r}}\right|_{(r_o,c_o)}\right]\dot{\tilde{r}}+\left[\left.\frac{\partial f}{\partial \dot{r}}\right|_{(r_o,c_o)}\right]\ddot{\tilde{r}}+\ldots \\
& +\left[\left.\frac{\partial f}{\partial c}\right|_{(r_o,c_o)}\right]\tilde{c}+\left[\left.\frac{\partial f}{\partial \dot{r}}\right|_{(r_o,c_o)}\right]\dot{\tilde{c}}+\left[\left.\frac{\partial f}{\partial \dot{r}}\right|_{(r_o,c_o)}\right]\ddot{\tilde{c}}+\ldots
\end{aligned}
$$

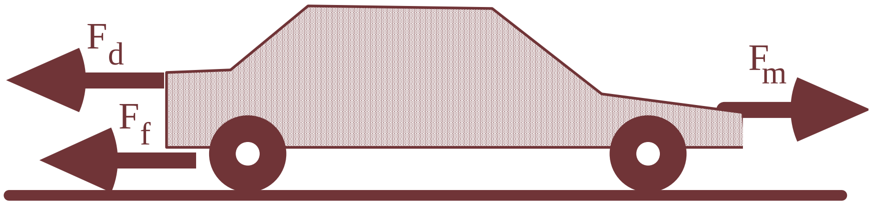
Note: Each of the terms in square brackets evaluates as a constant.

6) Rewrite the function defined in 5) in the standard ordinary differential equation form.

$$\left[\left.\frac{\partial f}{\partial \dot{r}}\right|_{(r_o,c_o)}\right]\ddot{\tilde{c}}+\left[\left.\frac{\partial f}{\partial \dot{r}}\right|_{(r_o,c_o)}\right]\dot{\tilde{c}}+\left[\left.\frac{\partial f}{\partial c}\right|_{(r_o,c_o)}\right]\tilde{c}=-\left[\left.\frac{\partial f}{\partial \dot{r}}\right|_{(r_o,c_o)}\right]\ddot{\tilde{r}}-\left[\left.\frac{\partial f}{\partial \dot{r}}\right|_{(r_o,c_o)}\right]\dot{\tilde{r}}-\left[\left.\frac{\partial f}{\partial r}\right|_{(r_o,c_o)}\right]\tilde{r}$$

Linearization Example

Aerodynamic drag is an inherently non-linear phenomenon that is commonly encountered in the control of vehicle speed. Examine the vehicle below traveling at velocity, v = 30 m/s.



The vehicle is pushed forward by the tractive force from the motor, F_m, while the vehicle is slowed down by the action of friction, F_f, and aerodynamic drag, F_d. We will model the tractive force as an input to the system. We will model the friction force as linear viscous friction and the aerodynamic drag as proportional to vehicle velocity squared.

$$F_m = F_m(t), \text{ tractive force input}$$
$$F_f = F_f(t) = K_f v, \text{ viscous friction force}$$
$$F_a = F_a(t) = K_a v^2, \text{ aerodynamic drag force}$$

Using Newton's law and the above free-body diagram to find the vehicle equation of motion yields a first order, nonlinear, ordinary differential equation.

$$m\dot{v} = F = F(F_m, v) = F_m(t) - K_f v - K_a v^2 \tag{1}$$

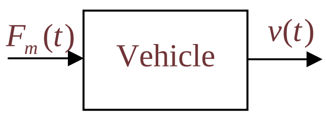
where: $m = 1{,}000$ kg, $K_f = 0.1$ N-sec/m, $K_a = 1.0$ N-sec$^2/m^2$. The model's non-linear differential equation becomes,

$$1000\dot{v} + 0.1v + v^2 = F_m \tag{2}$$

The applied forces are nonlinear in velocity and the aerodynamic drag term is the cause of this nonlinearity. Find a linearized system model valid near a velocity $v = v_O = 30$ m/sec.

Six Steps to Linearization…

1) Identify the input as tractive force F_m and the output as vehicle velocity v.



2) Express the non-linear model (2) as

$$f(F_m, v, \dot{v}) = 1000\dot{v} + 0.1v + v^{2,} - F_m = 0 \tag{3}$$

3) Define an equilibrium operating point for output velocity, $v = v_O = 30$ m/sec. Using (3) for equilibrium ($\dot{v} = 0$) at this velocity, $F_o = 0.1v_o + v_o^2 = 0.1(30) + 30^2 = 903\ N$. This operating point where $f(F_o, v_o, 0) = 0$ is

$$(F_m, v) = (F_o, v_o) = (903\ N,\ 30\ m/sec) \tag{4}$$

4) Performing the Taylor expansion of (3) yields,

$$f(F_m, v, \dot{v}) \cong f(F_o, v_o, 0) + \left[\frac{\partial f}{\partial F_m}\right]\Bigg|_{\substack{F_m=F_o \\ v=v_O}} (F_m - F_o) + \left[\frac{\partial F}{\partial v}\right]\Bigg|_{\substack{F_m=F_o \\ v=v_O}} (v - v_o) + \left[\frac{\partial F}{\partial \dot{v}}\right]\Bigg|_{\substack{F_m=F_o \\ v=v_O}} (\dot{v} - \dot{v}_o)$$

Now, $\left[\frac{\partial f}{\partial F_m}\right]\Bigg|_{\substack{F_m=F_o \\ v=v_O}} = -1$, $\left[\frac{\partial F}{\partial v}\right]\Bigg|_{\substack{F_m=F_o \\ v=v_O}} = 0.1 + 2v_o = 0.1 + 2(30) = 30.1$, and $\left[\frac{\partial F}{\partial \dot{v}}\right]\Bigg|_{\substack{F_m=F_o \\ v=v_O}} = 1000$

Substituting into the Taylor expansion yields,

$$f(F_m, v, \dot{v}) \cong 0 + (-1)(F_m - F_o) + (30.1)(v - v_o) + (1000)(\dot{v} - \dot{v}_o) \tag{5}$$

5) Changing the model's variables to those suggested by the Taylor expansion, $\tilde{F}_m = (F_m - F_o)$, $\tilde{v} = (v - v_o)$ and $\dot{\tilde{v}} = (\dot{v} - \dot{v}_o)$ gives the linearized function

$$f(\tilde{F}_m, \tilde{v}, \dot{\tilde{v}}) \cong (-1)\tilde{F}_m + 30.1\tilde{v} + 1000\dot{\tilde{v}} = 0 \tag{6}$$

6) Rewriting the linearized model to the standard linear ordinary differential equation form with the input on the right and the output terms on the left yields,

$$1000\dot{\tilde{v}} + 30.1\tilde{v} = \tilde{F}_m \tag{7}$$

The Six Linearization Steps Summarized:

1) Identify input and output variables.
2) Express non-linear differential equation in the form $f(r, \dot{r}, \ddot{r}, \ldots, c, \dot{c}, \ddot{c}, \ldots) = 0$
3) Find the equilibrium operating point $(r, c) = (r_o, c_o)$
4) Perform Taylor expansion neglect derivatives above first order.
5) Change variables: $\tilde{r} = (r - r_o)$, $\dot{\tilde{r}} = (\dot{r} - \dot{r}_o)$, $\ldots$, $\tilde{c} = (c - c_o)$, $\dot{\tilde{c}} = (\dot{c} - \dot{c}_o)$, $\ldots$
6) Rewrite result as a linear ODE in standard form.