Frequency Response Example:

For the Open-Loop Transfer Function $KGH = K\left[\frac{12}{(s+1)(s+2)(s+3)}\right]$

At zero frequency, this system has a DC (steady-state) gain of

$$GH(0) = \left(\frac{12}{1*2*3}\right) = 2$$

In a decibel (dB) scale, this gain is expressed.

$$GH(0) = 20\log_{10}(2) = 20(0.3010) = 6.02 \text{ dB}$$

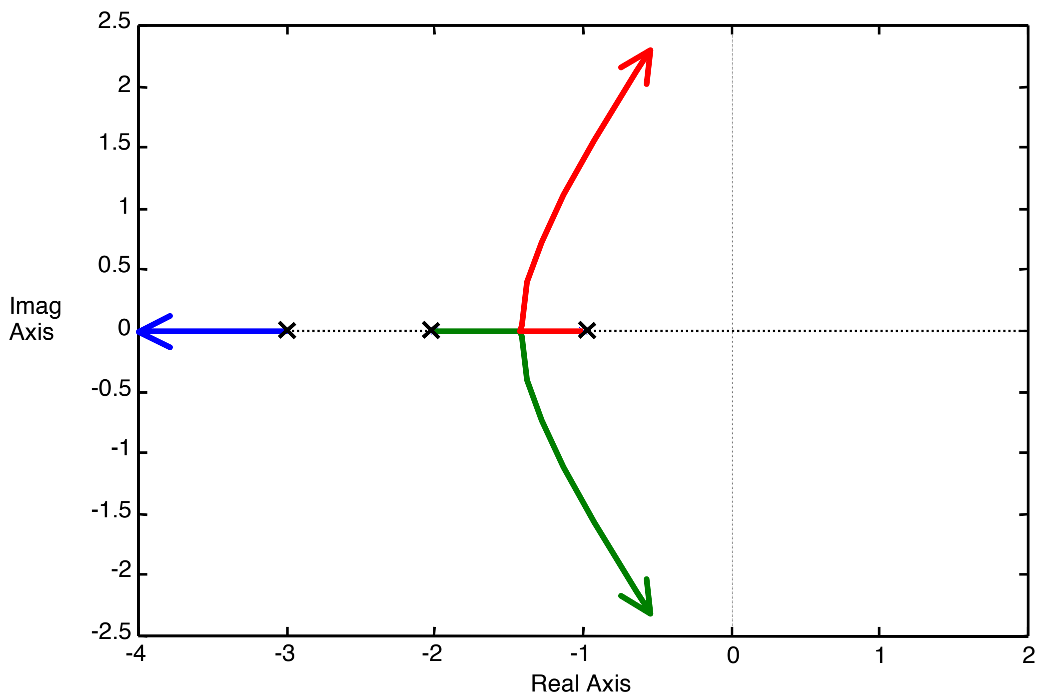
The two are equivalent but one is on a linear scale and the other on a logarithmic scale

Note: For the record, "deci-Bel" dB is the 10*log(power ratio)

$$dB = 10\log_{10}\left(\frac{P}{P_{ref}}\right) = 10\log_{10}\left(\frac{ky^2}{ky_{ref}^2}\right) = 10\log_{10}\left(\frac{y^2}{y_{ref}^2}\right) = 10\log_{10}\left[\left(\frac{y}{y_{ref}}\right)^2\right]$$

$$= 20\log_{10}\left(\frac{y}{y_{ref}}\right) = 20\log_{10}(y) \quad \text{for} \quad y_{ref} = 1 \quad \text{because} \quad \log_{10}(1) = 0$$

The Root Locus showing the closed-loop pole locations is



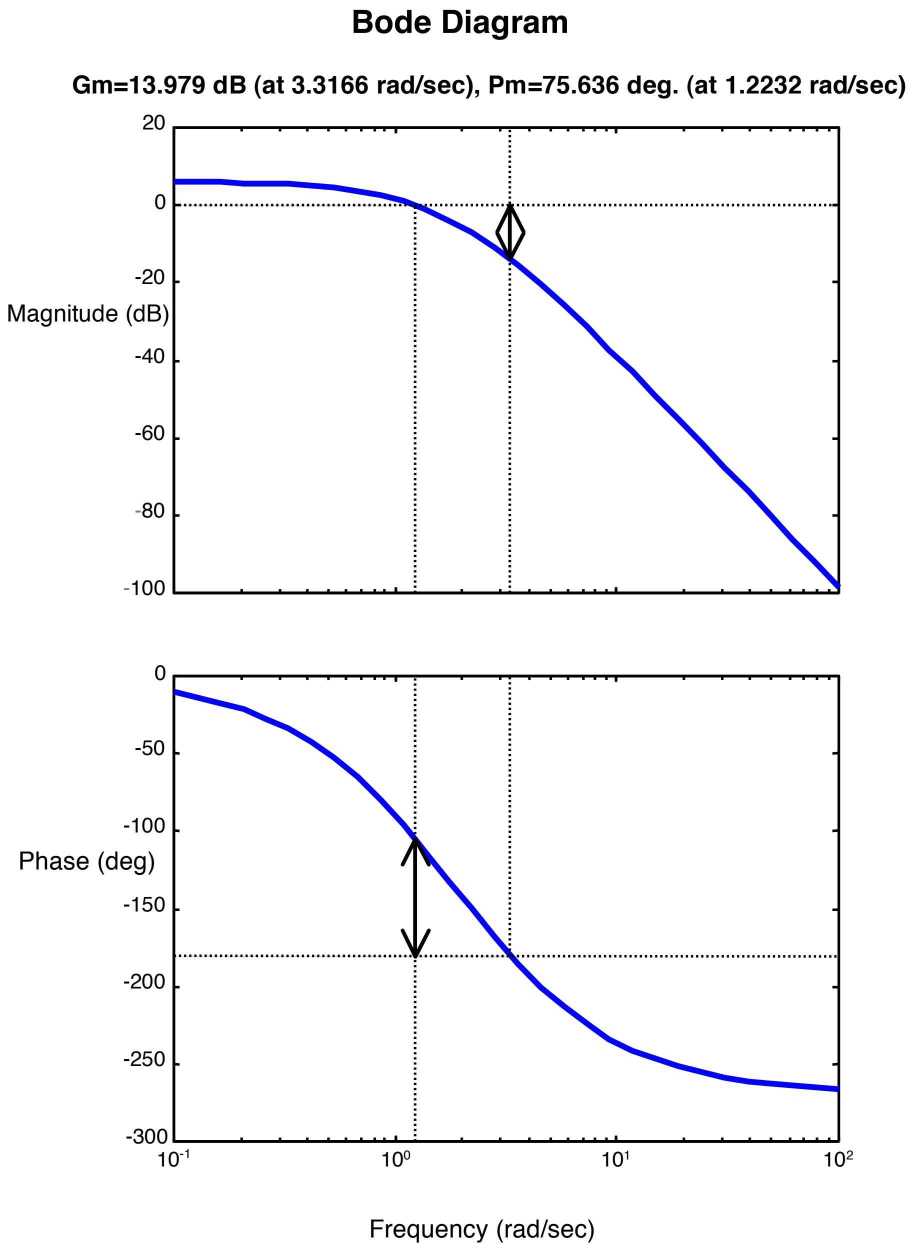
The Open-Loop Frequency Response (Bode Diagram) is plotted with the command

```
EDU» num=[12]; den=conv([1 1],conv([1 2], [1 3])); G=tf(num,den)
EDU» bode(G)
```

And the Gain and Phase margins are computed with the command

```
EDU» margin(G)
```

Yielding the plot...



This plot show a Gain Margin (GM) = 13.979 dB and a Phase Margin of 75.636 degrees.

The Gain Margin indicates that the control gain can be increased by 13.979 dB ($10^{(14/20)} = 5$) from $K = 1 = 0$dB before the system will go unstable.

The Phase Margin indicates that at the current gain (K = 1), the system can absorb an addition phase lag of 75.636 before it will go unstable.

For the system

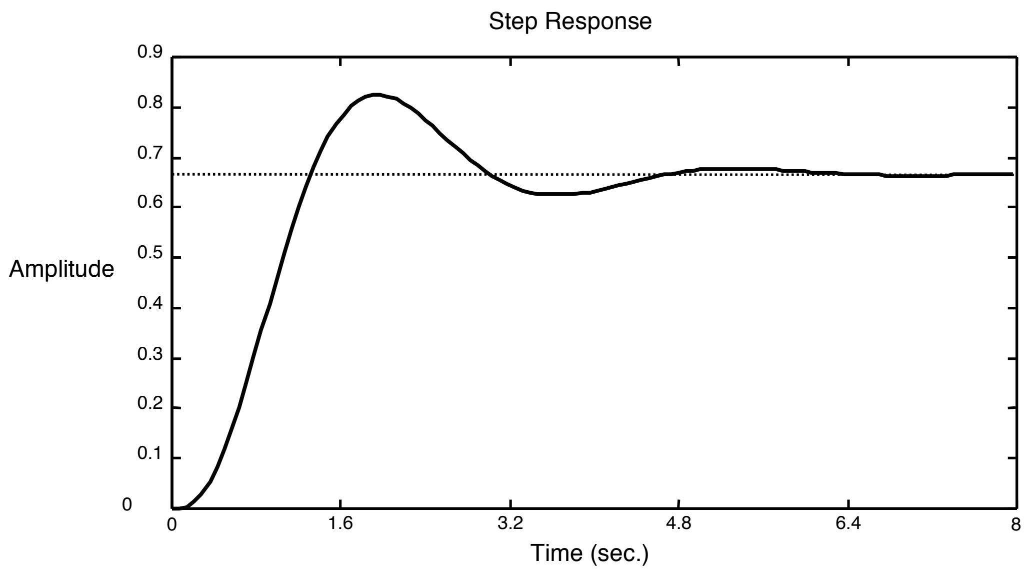
$$KGH = K\left[\frac{12}{(s+1)(s+2)(s+3)}\right]$$

the GM=14. dB and Phase Margin = 76 degrees with K=1indicate a very stable system.

The step response for the Closed-Loop system

$$T = \frac{12}{s^3 + 6s^2 + 11s + 18}$$

confirms this prediction.



Although the step response is stable, the steady state error is 33%. Could we reduce it by increasing the gain K from $K = 1\,(0\text{ dB})$ to $K = 2\,(6.02\text{ dB})$? Yes, just add 6.02 dB to the frequency response shift it up 6.02 dB at all frequencies. Let's plot it with the Matlab commands

```
EDU» num=[24]; den=[1 6 11 6]; G=tf(num,den);margin(G)
```

The new open-loop transfer function

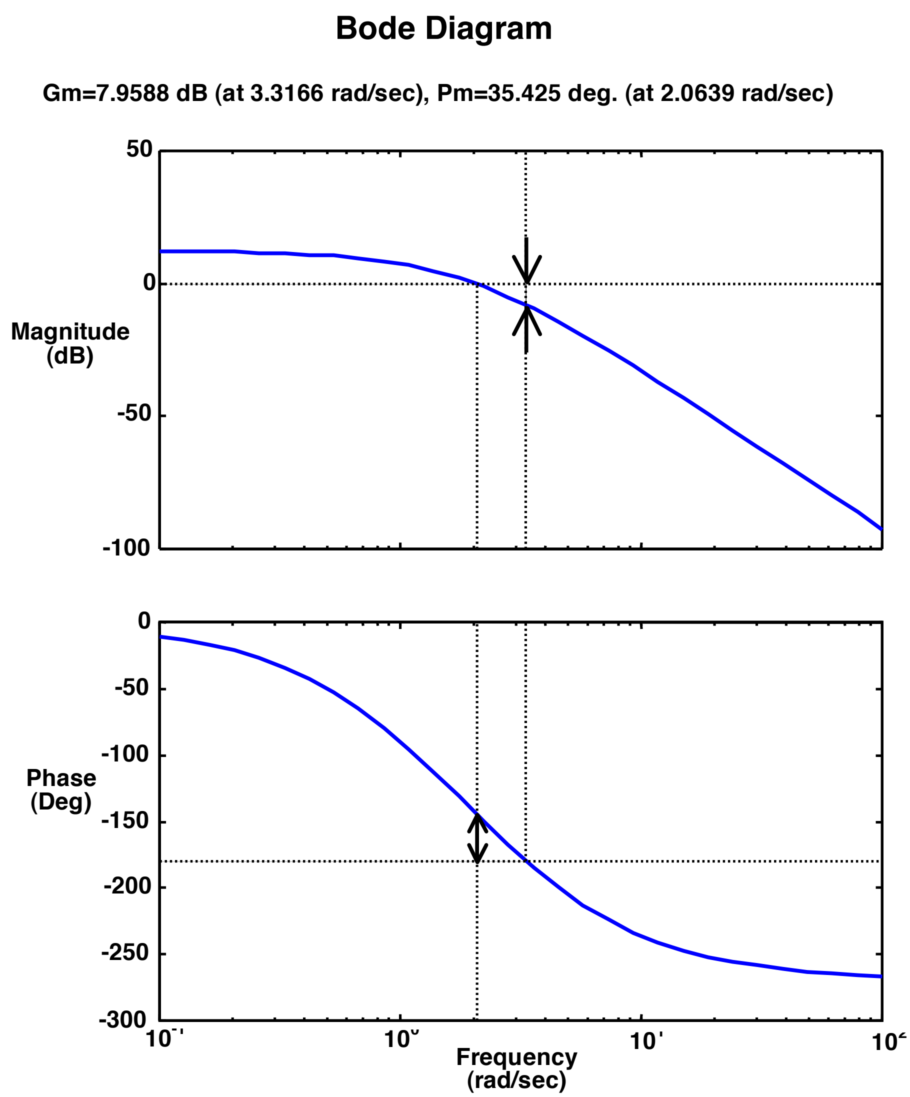
$$G_2(s) = \frac{24}{s^3 + 6s^2 + 11s + 6}$$

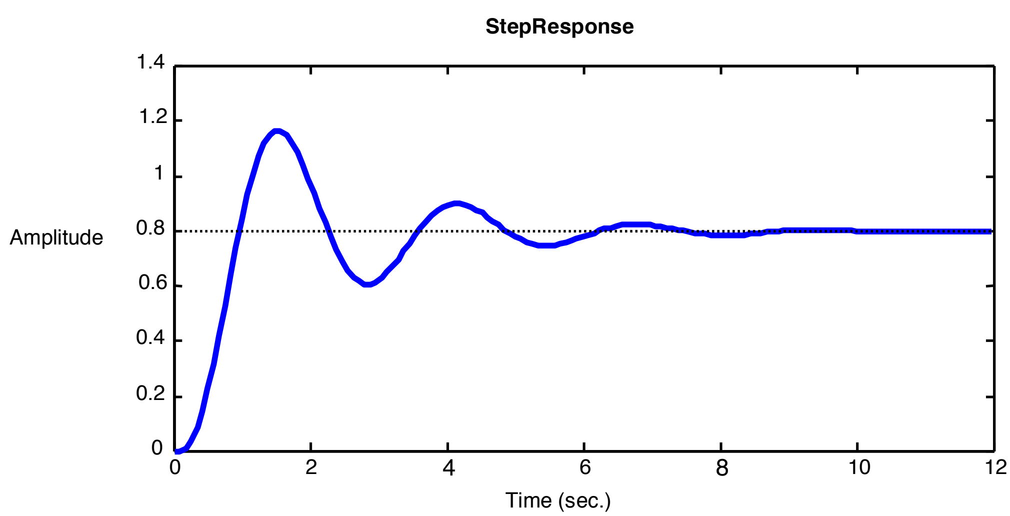
has a DC gain of $G_2(s) = 4 \quad (= 12\text{ dB})$ reducing the error to $e(\infty) = 1/(1+4) = 0.2 = 20\%$ but sacrifices stability with the new GM=8 dB and PM = 35 degrees.

The system's closed-loop transfer function

$$T_2 = \frac{24}{s^3 + 6s^2 + 11s + 30}$$

has the predicted, less stable response with better steady-state error.





More Advanced Control Design:

Our system still has inadequate steady-state accuracy – even at the high gain (K=2) where the limiting gain and phase margins are achieved. To increase steady-state accuracy, apply an integral control.

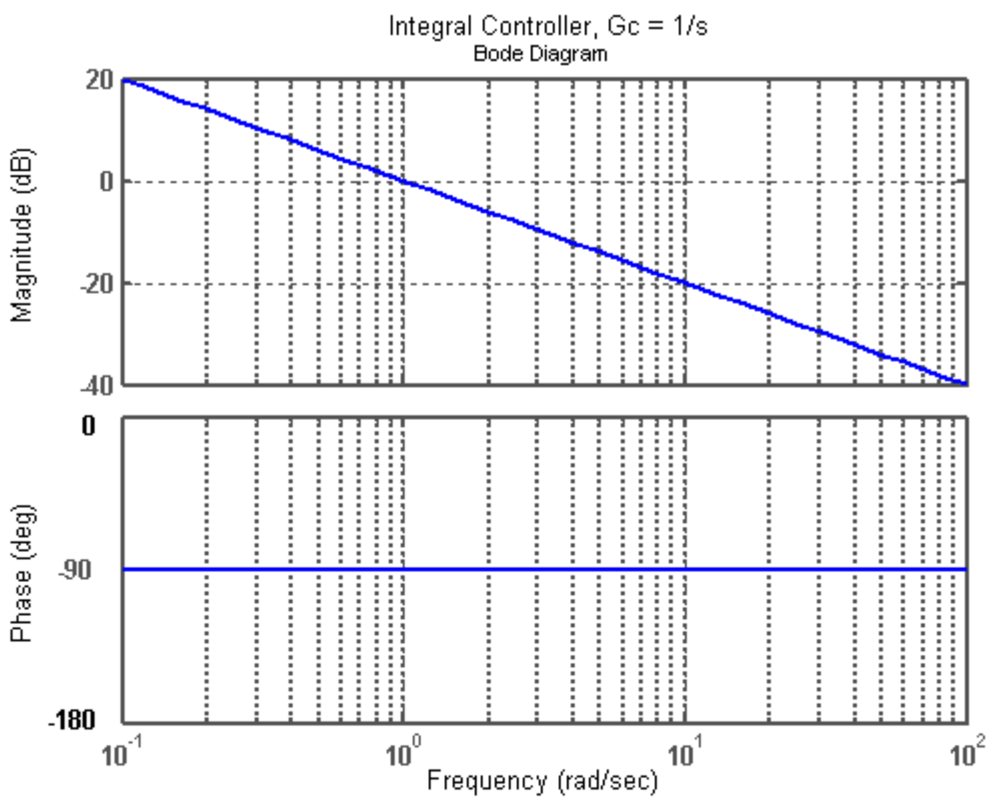
Integral Controller:

The transfer function of an integral controller is:

$$G_c(s) = K/s$$

The frequency response of this controller is plotted using the Matlab command

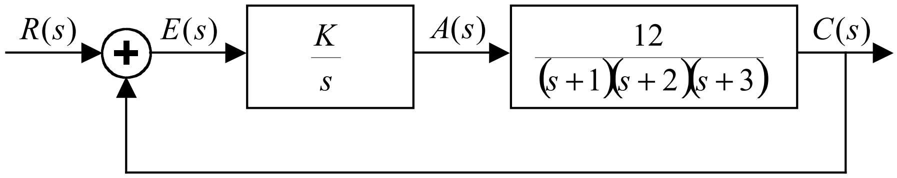
and is shown below for K=1.



Adding the magnitude and phase of the Integral controller to the magnitude and phase of the original controller can be done either graphically or analytically. The result is the Bode diagram of the open-loop transfer function

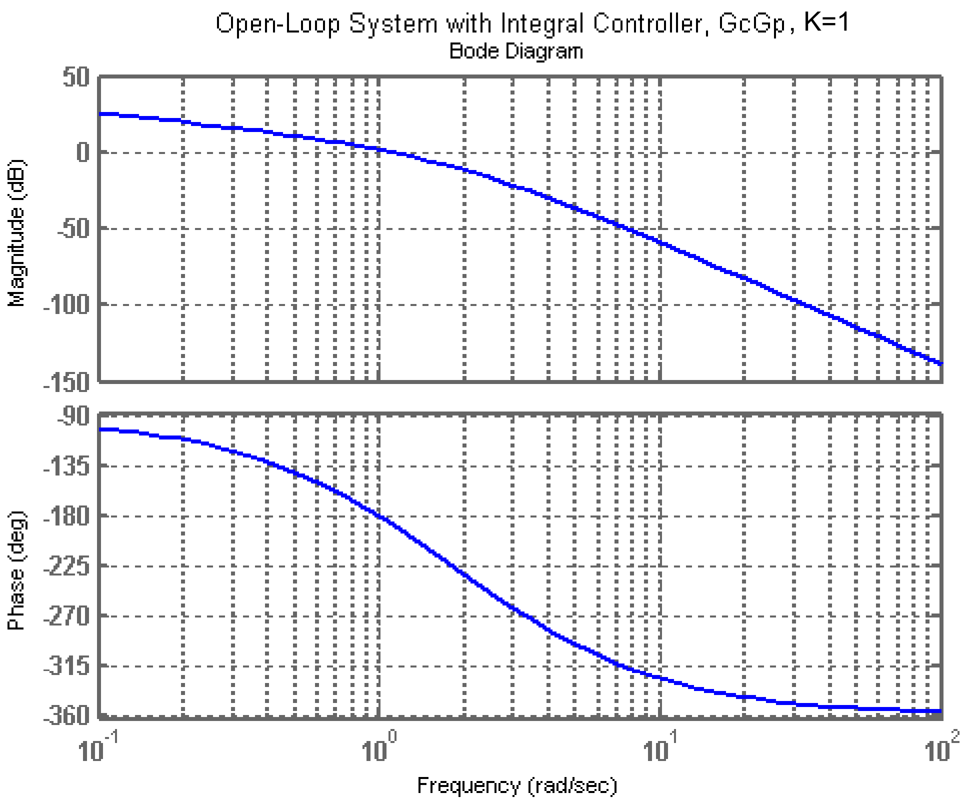
$$KGH = \left[\frac{K}{s}\right]\left[\frac{12}{(s+1)(s+2)(s+3)}\right] = K\left[\frac{12}{s(s+1)(s+2)(s+3)}\right]$$

of the system shown in the figure below.



Closed Loop system with Integral Controller

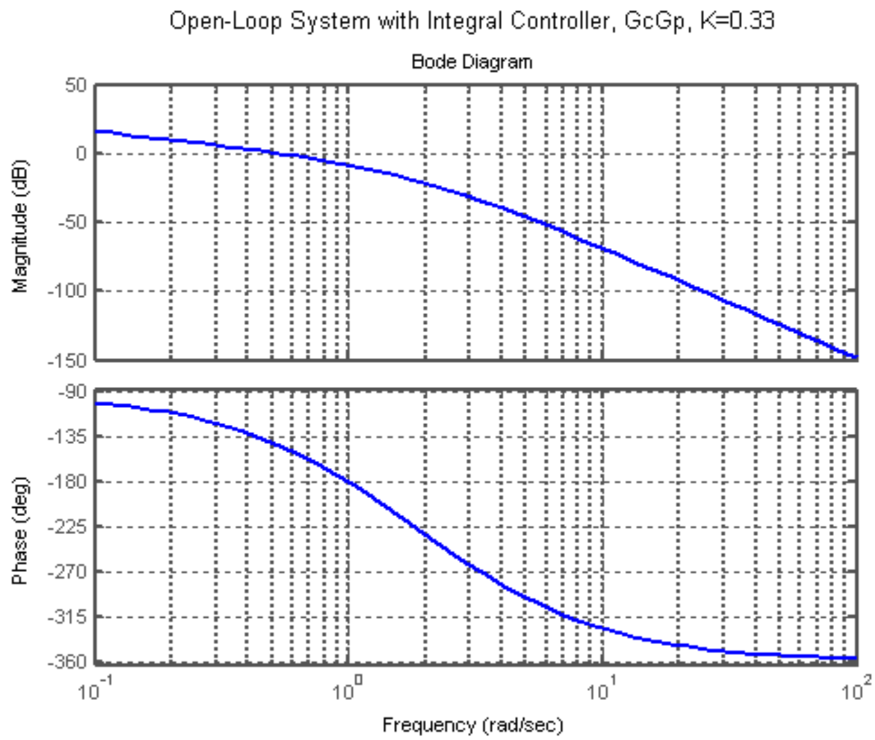
The Bode diagram is …



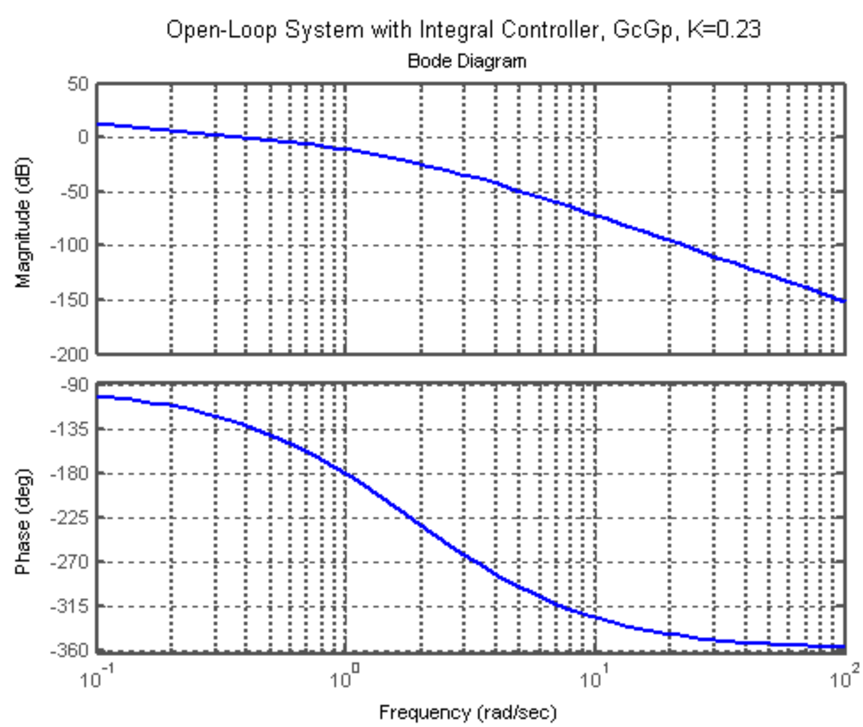
The resulting system has a Gain Margin, GM = -1.6 dB and the Phase Margin, PM = -7 degrees so it is unstable. To stabilize the system at an acceptable GM = 8 dB and PM = 45degrees requires a reduction in integral control gain by a factor of at least 8+1.6 = 9.6 dB to meet the gain margin requirement. The new control gain

$$K = -9.6\,\text{dB} = 10^{(-9.6/20)} = 0.33$$

The resulting bode diagram has a GM=8 dB and a PM=36 degrees and is shown below. This reduced gain now meets the gain margin (GM) requirement but does not achieve the required phase margin, PM = 45 degrees.



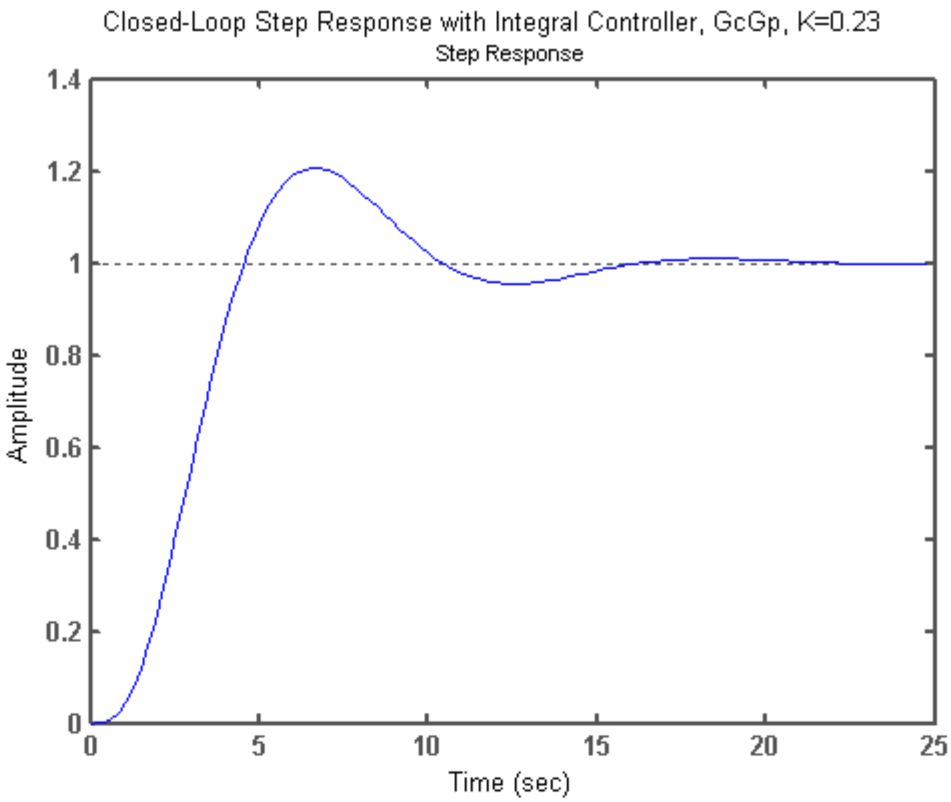
To achieve the require phase margin, decrease the gain K further. How much? Look at the Bode Diagram for K = 0.33and make an estimate. The 0 dB crossing point needs to move from about ω = 0.55 to about ω = 0.42. This requires a further magnitude adjustment of about 3 dB = 0.7. Try K=0.33*0.7 = 0.23. Matlab Computes this system's GM = 11.2 dB and PM = 48. degrees. The 0 dB crossing shows a bandwidth of about 0.4 rad/sec yielding a time constant of approximately $\tau = 2.5$ seconds.



The system integral controller design is $G_c = 0.23/s$ and has the closed-loop step response given by the closed-loop transfer function

$$T(s) = \frac{G_c G_p}{1 + G_c G_p} = \frac{(0.23*12)}{s^4 + 6s^3 + 11s^2 + 6s + (0.23*12)}$$

This system has a steady-state gain $T(0) = 1$ and no steady-state error at any gain because the open-loop system is type 1. The closed-loop system step response is given below.



The system's overall time constant is slower than predicted, T_s approximately 20 seconds yielding an equivalent time constant $\tau =$ about 5 seconds.

Proportional-Derivative Control:

A Proportional-Derivative (PD) control generates a "lead" action. This control will both increase the speed of response and further stabilize the system.

$$K_p + K_d s = K_d \left(s + {K_p}/{K_d} \right)$$

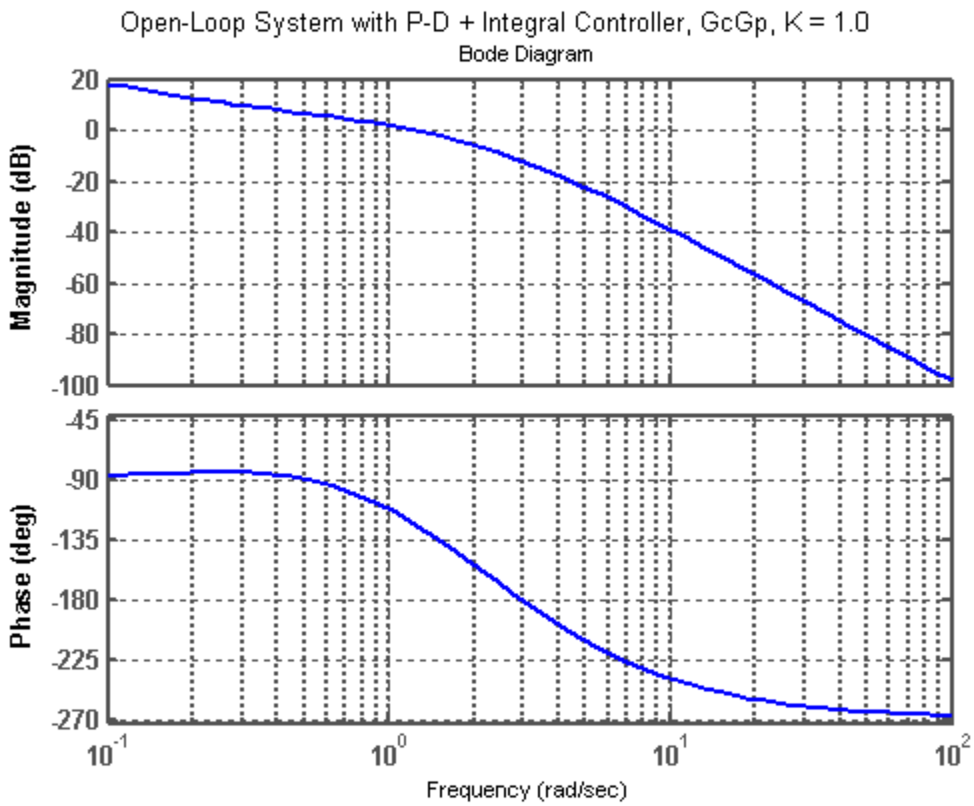
Propose a controller with a "zero" at $K_p / K_d = \omega = 0.4$ rad/sec. This additional control makes the controller transfer function

$$G_c = K \frac{(s + 0.4)}{s}$$

yielding the Open-Loop system transfer function for the P-D plus Integral controller

$$KGH = K\left[\frac{(s+0.4)}{s}\right]\left[\frac{12}{(s+1)(s+2)(s+3)}\right] = K\left[\frac{12(s+0.4)}{s(s+1)(s+2)(s+3)}\right]$$

The Bode diagram (open-loop frequency response) of this system with the modified controller for K=1 shows a decrease in phase lag about the zero "corner" frequency.

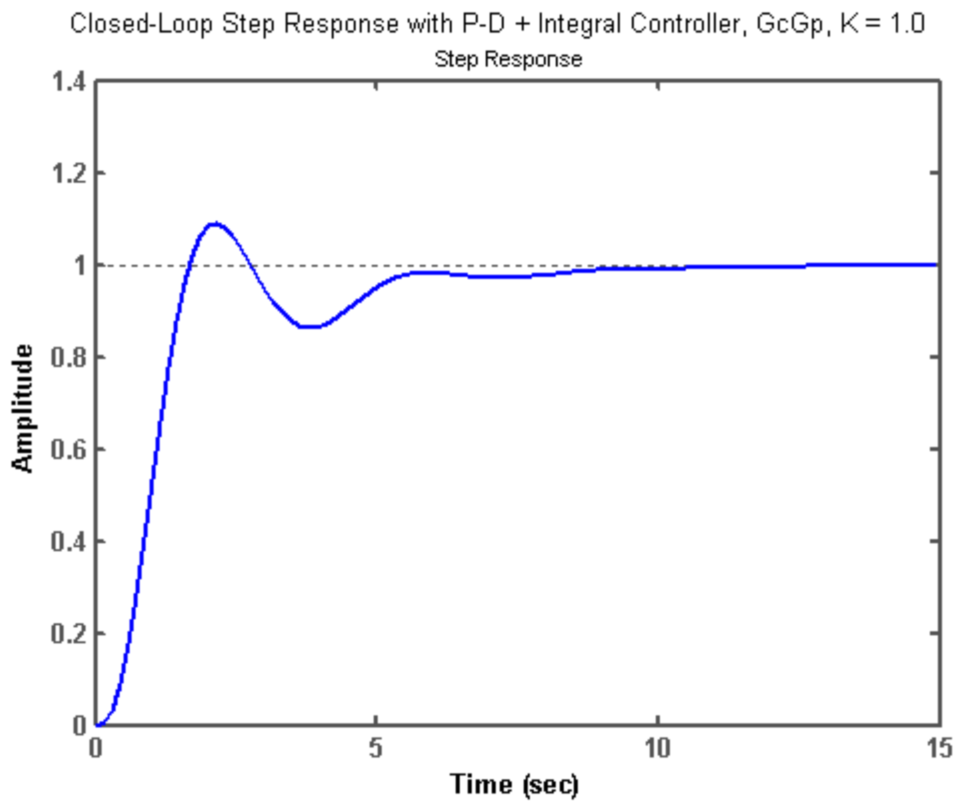


This new "P-D+I" controlled system has a gain margin GM = 11.9 dB and a phase margin PM = 55 degrees while retaining zero steady-state error. Furthermore, the bandwidth of the system $\omega =$ about 1.3 rad/sec (look at the 0 dB crossing) yielding an expected time constant $\tau =$ about 0.76 seconds and a settling time $T_s =$ about 3 seconds. It should be substantially faster than with only "I" action. The step response for the closed-loop system

$$T(s) = \frac{G_c G_p}{1+G_c G_p} = \frac{(1.0*12)(s+0.4)}{\mathrm{s}^4+6\,\mathrm{s}^3+11\,\mathrm{s}^2+6\mathrm{s}+(1.0*12)(s+0.4)}$$

$$= \frac{12\mathrm{s}+0.48}{\mathrm{s}^4+6\,\mathrm{s}^3+11\,\mathrm{s}^2+18\mathrm{s}+4.8}$$

verifies this expectation.



The result is a closed-loop system with settling time $T_S = 10$ seconds, corresponding to a time constant $\tau = 2.5$ sec. The system has about 10% overshoot because we designed it with plenty of gain and phase margin. This compares very favorably to the "Integral only" control that had a settling time of about 20 seconds. The "PD" action has decreased the time of response of our system substantially. Looks like a good controller.

You should see that although we have used the analytical model to explain what is happening and allow easy Matlab plotting of results, WE DID NOT NEED THE PLANT MODEL to do the design. In fact, the use of a Bode diagram allows the simple addition of the controller frequency response to a measured plant frequency response. Understanding with a transfer function model is important BUT you can simply add the controller transfer function to a measured plant transfer function if you do not have a transfer function model available.

A Final Note:

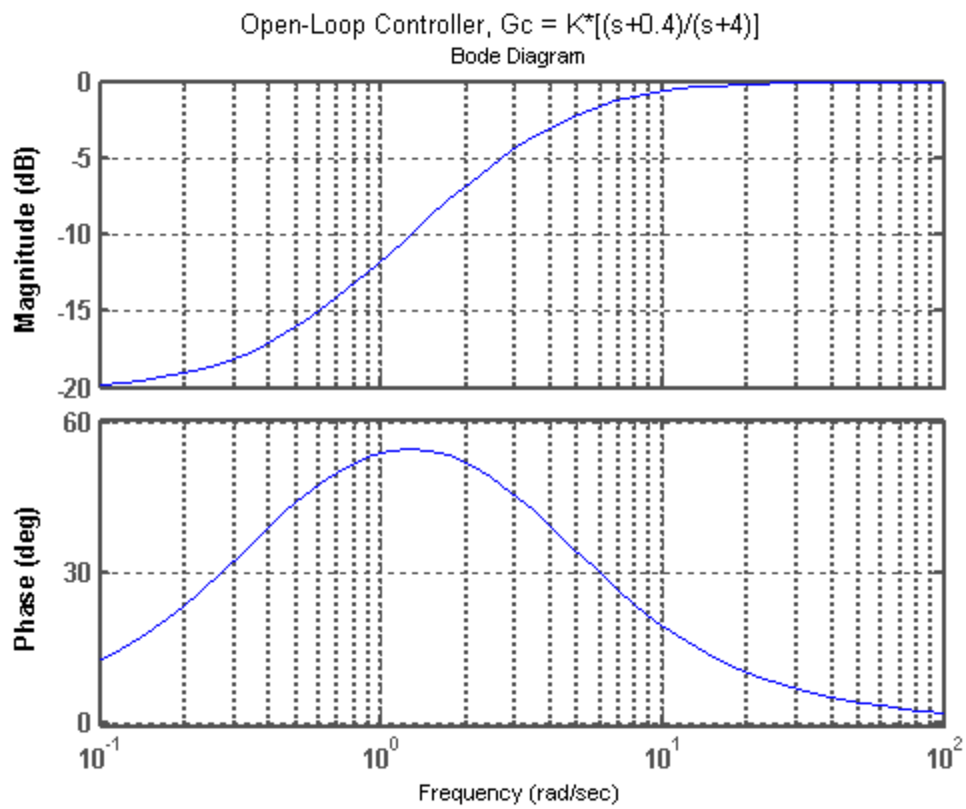
The addition of the "Proportional-Derivative" term to the Integral controller meant that we did not need to add a "pole" to the PD controller to implement it. If we wanted to add a "PD" term to the original system, we would have had to use something like

$$G_c = K_d \frac{\left(s + {K_p}/{K_d}\right)}{\left(s + \omega_{LP}\right)}$$

where ω_{LP} is some frequency above the frequency of the "zero" $K_p / K_d = \omega = 0.4$ rad/sec. I might try something like ω_{LP} = 4 rad/sec.

$$G_c = K \frac{(s + 0.4)}{(s + 4.0)}$$

which has the Bode diagram,



This diagram shows explicitly the phase lead generated by the pole-zero pair on the real axis with the pole to the left (high frequency $\omega = 4.0$ rad/sec) of the zero (low frequency $\omega = 0.4$ rad/sec). This figure also shows the 20 dB = 10 (linear) magnitude reduction that results at zero frequency normally causing reduced system steady-state accuracy through the resulting larger steady-state error.