Simple mechanical examples

- We want mass to stay at x=0, but wind gave some initial speed (F(t)=0). What will happen?

\[
\begin{align*}
X(s) &= \frac{1}{s^2} & \text{simple mechanical case} \\
X(s) &= \frac{1}{s^2 + KB} & \text{electromechanical case}
\end{align*}
\]

- How to characterize different behaviors with TF?

Stability

- Utmost important specification in control design!
- Unstable systems have to be stabilized by feedback.
- Unstable closed-loop systems are useless.
  - What happens if a system is unstable?
    - may hit mechanical/electrical “stops” (saturation)
    - may break down or burn out
What happens if a system is unstable?

Tacoma Narrows Bridge (July 1-Nov.7, 1940)

Wind-induced vibration

Collapsed!

2008...

Mathematical definitions of stability

- **BIBO (Bounded-Input-Bounded-Output) stability**
  
  Any bounded input generates a bounded output.

  \[ u(t) \rightarrow \text{BIBO stable system} \rightarrow y(t) \]

  - Initial Conditions (ICs) = 0

- **Asymptotic stability**
  
  Any ICs generates \( y(t) \) converging to zero.

  \[ u(t) = 0 \rightarrow \text{Asymptotic stable system} \rightarrow y(t) \]

  - ICs

Some terminologies

- **Zero** : roots of \( n(s) \)
  
  \( (Zeros \ of \ G) = \pm 1 \)

- **Pole** : roots of \( d(s) \)
  
  \( (Poles \ of \ G) = -2, \pm j \)

- **Characteristic polynomial** : \( d(s) \)

- **Characteristic equation** : \( d(s)=0 \)

Stability condition in s-domain

(Proof omitted, and not required)

For a system represented by a transfer function \( G(s) \),

\[ \text{system is BIBO stable} \]

- **All the poles of } G(s) \text{ are in the open left half of the complex plane.}**

\[ \text{system is asymptotically stable} \]
“Idea” of stability condition

Example \( y'(t) + \alpha y(t) = u(t), \ y(0) = y_0 \)

\[ sY(s) - y(0) + \alpha Y(s) = U(s) \]

\[ Y(s) = \frac{1}{s + \alpha} (U(s) + y(0)) \]

Asym. Stability: 
\[ y(t) = \mathcal{L}^{-1} \left\{ Y(s) \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{s + \alpha} y(0) \right\} = e^{-\alpha t} y(0) \rightarrow 0 \Rightarrow \text{Re}(\alpha) > 0 \]

BIBO Stability: 
\[ y(t) = \mathcal{L}^{-1} \left\{ Y(s) \right\} = \mathcal{L}^{-1} \left\{ G(s)U(s) \right\} = \int_0^t e^{-\alpha t} u(t-\tau)d\tau = \int_0^t e^{-\alpha \tau} u(t-\tau)d\tau \]

\[ |y(t)| \leq \int_0^t e^{-\alpha \tau} |u(t-\tau)|d\tau \leq \int_0^t e^{-\alpha \tau} |d\tau| \cdot u_{\text{max}} \]

Bounded if \( \text{Re}(\alpha) > 0 \)

Second order impulse response – Underdamped and Undamped

Changing \( \zeta / \text{Fixed } \omega_n \)

\[ h(t) = \frac{\omega_n}{\sqrt{1 - \zeta^2}} e^{-\zeta \omega_n \sqrt{1 - \zeta^2} t} 1(t) \]

Second order impulse response – Underdamped and Undamped

Changing \( \zeta / \text{Fixed } \omega_n \)

\[ h(t) = \frac{\omega_n}{\sqrt{1 - \zeta^2}} e^{-\omega_n \sqrt{1 - \zeta^2} t} 1(t) \]
Second order impulse response –
Underdamped and Undamped

Changing $\zeta$ / Fixed $\omega_n$

$h(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta \omega_n \sqrt{1-\zeta^2}t} \sin\left(\omega_n \sqrt{1-\zeta^2} t\right)$

Remarks on stability

- For a general system (nonlinear etc.), BIBO stability condition and asymptotic stability condition are different.
- For linear time-invariant (LTI) systems (to which we can use Laplace transform and we can obtain a transfer function), the conditions happen to be the same.
- In this course, we are interested in only LTI systems, we use simply “stable” to mean both BIBO and asymptotic stability.

Remarks on stability (cont’d)

- **Marginally stable** if
  - $G(s)$ has no pole in the open RHP (Right Half Plane), &
  - $G(s)$ has at least one simple pole on $j\omega$-axis, &
  - $G(s)$ has no multiple poles on $j\omega$-axis.

\[
G(s) = \frac{1}{s(s^2 + 4)(s + 1)} \\
G(s) = \frac{1}{s(s^2 + 4)^2(s + 1)}
\]

- **Unstable** if a system is neither stable nor marginally stable.

Examples

- **Repeated poles**

\[
\mathcal{L}^{-1}\left[\frac{2\omega_n}{(s^2 + \omega_n^2)^2}\right] = t \sin \omega t \\
\mathcal{L}^{-1}\left[\frac{s^2 - \omega_n^2}{(s^2 + \omega_n^2)^2}\right] = t \cos \omega t \\
\ldots = t^2 \sin \omega t \\
\ldots = t^2 \cos \omega t
\]

- **Does marginal stability imply BIBO stability?**

- **TF:**

\[G(s) = \frac{2s}{(s^2 + 1)}\]

- **Pick**

\[u(t) = \sin t \quad \mathcal{L}^{-1}[u(t)] = \frac{1}{s^2 + 1}\]

- **Output**

\[
\mathcal{L}^{-1}\left[Y(s) = G(s)U(s) = \frac{2s}{(s^2 + 1)^2}\right] = t \sin t
\]
Feedback Technique

Positive Feedback

K will depend on the distance between the guitar and the amplifier.

Stability summary

Let \( s_i \) be poles of \( G \).
Then, \( G \) is ...

- **(BIBO, asymptotically) stable** if \( \text{Re}(s_i)<0 \) for all \( i \).
- **marginally stable** if
  - \( \text{Re}(s_i)\leq0 \) for all \( i \), and
  - simple root for \( \text{Re}(s_i)=0 \)
- **unstable** if
  it is neither stable nor marginally stable.

Mechanical examples: revisited

- \( F(s) = \frac{1}{s^2} \)
  - **Poles = stable?**
- \( F(s) = \frac{1}{s^2+K} \)
  - \( K \) depends on the distance between the guitar and the amplifier.
- \( F(s) = \frac{1}{s^2+B} \)
  - \( B \) depends on the distance between the guitar and the amplifier.
- \( F(s) = \frac{1}{s^2+B+K} \)
  - \( K \) depends on the distance between the guitar and the amplifier.
### Examples

<table>
<thead>
<tr>
<th>( G(s) )</th>
<th>Stable/marginally stable/unstable</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{5(s + 2)}{(s + 1)(s^2 + s + 1)} )</td>
<td>?</td>
</tr>
<tr>
<td>( \frac{5(-s + 2)}{(s + 1)(s^2 + s + 1)} )</td>
<td>?</td>
</tr>
<tr>
<td>( \frac{5}{(s + 1)(s^2 + s + 1)} )</td>
<td>?</td>
</tr>
<tr>
<td>( \frac{(s - 2)(s^2 + 3)}{s^2 + 3} )</td>
<td>?</td>
</tr>
<tr>
<td>( \frac{s^2 + 3}{(s + 1)(s^2 - s + 1)} )</td>
<td>?</td>
</tr>
<tr>
<td>( \frac{1}{(s + 1)(s^2 + 1)^2} )</td>
<td>?</td>
</tr>
<tr>
<td>( \frac{1}{(s^2 - 1)(s + 1)} )</td>
<td>??</td>
</tr>
</tbody>
</table>

### Summary and Exercises

- **Stability for LTI systems**
  - (BIBO and asymptotically) stable, marginally stable, unstable
  - Stability for \( G(s) \) is determined by poles of \( G \).
- **Next**
  - Routh-Hurwitz stability criterion to determine stability without explicitly computing the poles of a system.
- **Exercises**
  - Solve examples in the previous slide.