ME451: Control Systems

Lecture 7
Linearization, time delays

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Course roadmap

Modeling
- ✓ Laplace transform
- ✓ Transfer function
- ✓ Models for systems
  - ✓ electrical
  - ✓ mechanical
  - electromechanical
- ✓ Block diagrams
- ➤ Linearization

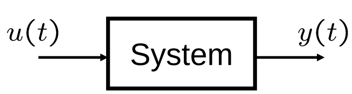
Analysis
- Time response
  - Transient
  - Steady state
- Frequency response
  - Bode plot
- Stability
  - Routh-Hurwitz
  - Nyquist

Design
- Design specs
- Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

(Matlab simulations &) laboratories

What is a linear system?

- A system having ***Principle of Superposition***



$$\left.\begin{array}{l} u_1(t) \rightarrow y_1(t) \\ u_2(t) \rightarrow y_2(t) \end{array}\right\} \Rightarrow \alpha_1 u_1(t) + \alpha_2 u_2(t) \rightarrow \alpha_1 y_1(t) + \alpha_2 y_2(t)$$
$$\forall \alpha_1, \alpha_2 \in \mathbb{R}$$

A nonlinear system does not satisfy the principle of superposition.

Linear systems

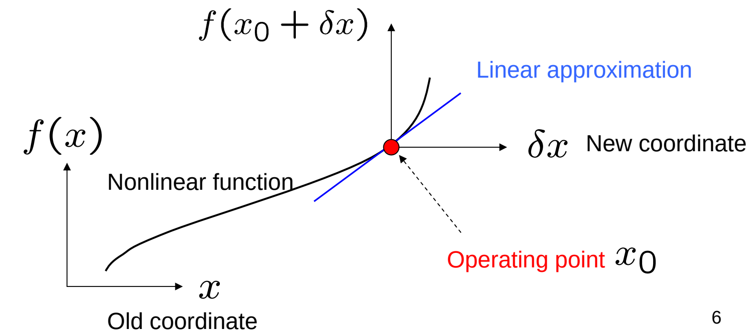
- Easier to understand and obtain solutions
- Linear ordinary differential equations (ODEs),
  - Homogeneous solution and particular solution
  - Transient solution and steady state solution
  - Solution caused by initial values, and forced solution
- Add many simple solutions to get more complex ones (use superposition!)
- Easy to check the Stability of stationary states (Laplace Transform)

Why linearization?

- Real systems are inherently nonlinear. (Linear systems do not exist!) *Ex.* f(t)=Kx(t), v(t)=Ri(t)
- TF models are only for linear time-invariant (LTI) systems.
- Many control analysis/design techniques are available for linear systems.
- Nonlinear systems are difficult to deal with mathematically.
- Often we linearize nonlinear systems before analysis and design. How?

How to linearize it?

- Nonlinearity can be approximated by a linear function for small deviations δx around an operating point x_0
- Use a Taylor series expansion



Linearization

- Nonlinear system: $\dot{x} = f(x,u)$
- Let u_0 be a nominal input and let the resultant state be x_0
- Perturbation: $u(\cdot) = u_o(\cdot) + \delta u(\cdot)$
- Resultant perturb: $x(\cdot) = x_o(\cdot) + \delta x(\cdot)$
- Taylor series expansion:

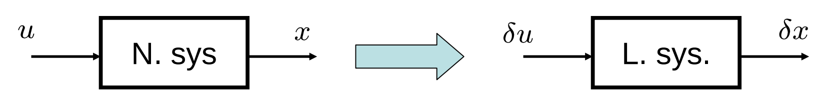
$$f(x,u) = f(x_0,u_0) + \frac{\partial f(x,u)}{\partial x}\Big|_{\substack{x=x_0\\u=u_0}} \delta x + \frac{\partial f(x,u)}{\partial u}\Big|_{\substack{x=x_0\\u=u_0}} \delta u + \underbrace{\text{H.O.T.}}_{\approx 0}$$

Linearization (cont.)

$$\dot{x}_0 + \delta\dot{x} = f(x_0,u_0) + \frac{\partial f(x,u)}{\partial x}\Big|_{\substack{x=x_0\\u=u_0}} \delta x + \frac{\partial f(x,u)}{\partial u}\Big|_{\substack{x=x_0\\u=u_0}} \delta u$$

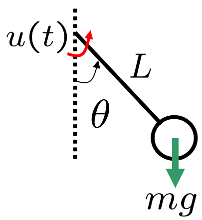
notice that $\dot{x}_0 = f(x_0,u_0)$; hence

$$\delta\dot{x} = \frac{\partial f(x,u)}{\partial x}\Big|_{\substack{x=x_0\\u=u_0}} \delta x + \frac{\partial f(x,u)}{\partial u}\Big|_{\substack{x=x_0\\u=u_0}} \delta u$$



Linearization of a pendulum model

- Motion of the pendulum



$$mL^2\ddot{\theta}(t) + mgL\sin\theta(t) = u(t)$$

$$\ddot{\theta}(t) + \underbrace{\frac{g\sin\theta(t)}{L} - \frac{u(t)}{mL^2}}_{f(\theta,u)} = 0$$

- Linearize it at $\theta_0 = \pi$
- Find u_0 $\quad \ddot{\pi} + \frac{g\sin\pi}{L} - \frac{u_0}{mL^2} = 0 \rightarrow u_0 = 0$
- New coordinates:

$$\theta = \theta_0 + \delta\theta = \pi + \delta\theta$$
$$u = u_0 + \delta u = 0 + \delta u$$

Linearization of a pendulum model (cont')

- Taylor series expansion of $f(\theta,u)$ at $\theta = \pi, u = 0$

$$\frac{\partial f(\theta,u)}{\partial\theta}\Big|_{\substack{\theta=\pi\\u=0}} = \frac{g\cos\theta}{L}\Big|_{\theta=\pi} = -\frac{g}{L}$$
$$\frac{\partial f(\theta,u)}{\partial u}\Big|_{\substack{\theta=\pi\\u=0}} = -\frac{1}{mL^2}$$

$$\delta\ddot{\theta} + \frac{\partial f(\theta,u)}{\partial\theta}\Big|_{\substack{\theta=\pi\\u=0}}\delta\theta + \frac{\partial f(\theta,u)}{\partial u}\Big|_{\substack{\theta=\pi\\u=0}}\delta u = 0$$

$$\boxed{\delta\ddot{\theta} - \frac{g}{L}\delta\theta - \frac{1}{mL^2}\delta u = 0}$$

Time delay transfer function

- TF derivation

$$y(t) = x(t - T_d) \ (T_d\text{: delay time})$$

$$\xrightarrow{\mathcal{L}} Y(s) = e^{-T_d s}X(s) \Rightarrow \frac{Y(s)}{X(s)} = e^{-T_d s}$$

(Memorize this!)

- The more time delay is, the more difficult to control (Imagine that you are controlling the temperature of your shower with a very long hose. You will either get burned or frozen!)

Summary and Exercises

- Modeling of
  - Nonlinear systems
  - Systems with time delay
- Next
  - Modeling of DC motors
- Exercises
  - Linearize the pendulum model at $\pi/4$