Lecture 3
Solution to ODEs via Laplace transform

Dr. Jongeun Choi
Department of Mechanical Engineering
Michigan State University

Laplace transform (review)

- One of most important math tools in the course!
- Definition: For a function \( f(t) \) (\( f(t)=0 \) for \( t<0 \)),
  \[
  F(s) = \mathcal{L}\{f(t)\} := \int_{0}^{\infty} f(t)e^{-st}dt
  \]
  (s: complex variable)
- We denote Laplace transform of \( f(t) \) by \( F(s) \).

An advantage of Laplace transform

- We can transform an ordinary differential equation (ODE) into an algebraic equation (AE).

\[
\begin{align*}
\text{ODE} & \quad \xrightarrow{\mathcal{L}} \quad \text{AE} \\
\text{Solution to ODE} & \quad \xrightarrow{\mathcal{L}^{-1}} \quad \text{Partial fraction expansion}
\end{align*}
\]
Example 1

ODE with initial conditions (ICs)

\[
\frac{d^2y(t)}{dt^2} + 3 \frac{dy(t)}{dt} + 2y(t) = 5u_s(t), \quad y(0) = -1, \quad y'(0) = 2
\]

1. Laplace transform

\[
L \{y''(t)\} - sy(0) - y'(0) + 3 \{sY(s) - y(0)\} + 2Y(s) = \frac{5}{s}
\]

\[
L \{y'(t)\} = \frac{5}{s}
\]

\[
Y(s) = \frac{-s^2 - s + 5}{s(s + 1)(s + 2)}
\]

Example 1 (cont’d)

2. Partial fraction expansion

\[
Y(s) = \frac{-s^2 - s + 5}{s(s + 1)(s + 2)} = \frac{A}{s} + \frac{B}{s + 1} + \frac{C}{s + 2}
\]

Multiply both sides by \(s\) & let \(s\) go to zero:

\[
sY(s)\big|_{s=0} = A + s \frac{B}{s + 1}\big|_{s=0} + s \frac{C}{s + 2}\big|_{s=0} \quad \Rightarrow \quad A = sY(s)\big|_{s=0} = \frac{5}{2}
\]

Similarly,

\[
B = (s + 1)Y(s)\big|_{s=-1} = \cdots = -5
\]

\[
C = (s + 2)Y(s)\big|_{s=-2} = \cdots = \frac{3}{2}
\]

Example 1 (cont’d)

3. Inverse Laplace transform

\[
L^{-1} \left\{ Y(s) = \frac{A}{s} + \frac{B}{s + 1} + \frac{C}{s + 2} \right\}
\]

\[
y(t) = \left( \frac{5}{2A} + \frac{(-5)}{B}e^{-t} + \frac{3}{2C}e^{-2t} \right) u_s(t)
\]

If we are interested in only the final value of \(y(t)\), apply Final Value Theorem:

\[
\lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s) = \lim_{s \to 0} \frac{-s^2 - s + 5}{(s + 1)(s + 2)} = \frac{5}{2}
\]
Example 2

\[ \ddot{y}(t) - y(t) = t, \quad y(0) = 1, \dot{y}(0) = 1 \]

- **S1**
  \[ s^2Y(s) - sy(0) - \dot{y}(0) - Y(s) = \frac{1}{s^2}, \]

- **S2**
  \[
  Y(s) = \frac{1}{s-1} + \frac{1}{s^2(s^2 - 1)} \\
  = \frac{1}{s-1} + \frac{1}{s^2(s^2 - 1)} \\
  = \frac{1}{s-1} + \frac{1}{2s-1} - \frac{1}{2s+1} - \frac{1}{s^2}
  \]

- **S3**
  \[ y(t) = \mathcal{L}^{-1}(Y(s)) = \left[ \frac{3}{2}e^t - \frac{1}{2}e^{-t} - t \right] u_s(t) \]

In this way, we can find a rather complicated solution to ODEs easily by using Laplace transform table!

Example: Newton’s law

\[ M \frac{d^2x(t)}{dt^2} = f(t) \quad \Rightarrow \quad x(t) \]

We want to know the trajectory of \( x(t) \). By Laplace transform,

\[ M \left( s^2X(s) - sx(0) - x'(0) \right) = F(s) \]

\[ X(s) = \frac{1}{Ms^2}F(s) + \frac{x(0)}{s} + \frac{x'(0)}{s^2} \]

(Total response) = (Forced response) + (Initial condition response)

\[ x(t) = \mathcal{L}^{-1} \left[ \frac{1}{Ms^2}F(s) \right] + x(0)u_s(t) + x'(0)tus(t) \]

EX. Air bag and accelerometer

- Tiny MEMS accelerometer
  - Microelectromechanical systems (MEMS)

(Pictures from various websites)
We would like to know how $y(t)$ moves when unit step $f(t)$ is applied with zero ICs.

By Newton’s law

\[
M \frac{d^2}{dt^2}(x(t) + y(t)) = -b \frac{dy(t)}{dt} - ky(t)
\]

\[
M_0 \frac{d^2}{dt^2} x(t) = f(t)
\]

\[
MY''(t) + By'(t) + Ky(t) = -\frac{M}{M_0} f(t)
\]

\[
\mathcal{L} \{ Y(s) \} = \frac{M}{M_0} \left( \frac{1}{s^2 + bs + k} \right) = \frac{1}{s} \left( \frac{1}{s^2 + \left(\frac{b}{M}\right)s + \left(\frac{k}{M}\right)} \right)
\]

Suppose that $b/M=3$, $k/M=2$ and $M_0=1$.

Partial fraction expansion

\[
Y(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{2s} + \frac{1}{s+1} + \frac{1}{2(s+2)}
\]

Inverse Laplace transform

\[
y(t) = \left( \frac{1}{2} e^{-t} - \frac{1}{2} e^{-2t} \right) u(t)
\]

Solution procedure to ODEs
1. Laplace transform
2. Partial fraction expansion
3. Inverse Laplace transform

Next, modeling of physical systems using Laplace transform

Exercises

- Derive the solution to the accelerometer problem.
- E2.4 of the textbook in page 135.