Lecture 3
Solution to ODEs via Laplace transform

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Laplace transform (review)

- One of most important math tools in the course!
- Definition: For a function \( f(t) \) (\( f(t) = 0 \) for \( t < 0 \)),
  \[
  F(s) = \mathcal{L}\{f(t)\} := \int_{0}^{\infty} f(t)e^{-st}dt
  \]
  \( (s: \text{complex variable}) \)

  \[
  f(t) \quad t \quad F(s)
  \]

- We denote Laplace transform of \( f(t) \) by \( F(s) \).

An advantage of Laplace transform

- We can transform an ordinary differential equation (ODE) into an algebraic equation (AE).

  \[
  \text{ODE} \quad \mathcal{L} \quad \text{AE}
  \]

  \[
  \text{Solution to ODE} \quad \mathcal{L}^{-1} \quad \text{Partial fraction expansion}
  \]
Example 1

ODE with initial conditions (ICs)
\[ \frac{d^2y(t)}{dt^2} + 3 \frac{dy(t)}{dt} + 2y(t) = 5u_s(t), \ y(0) = -1, \ y'(0) = 2 \]

1. Laplace transform

\[ \mathcal{L}\{y''(t)\} \quad \mathcal{L}\{y'(t)\} \]

Properties of Laplace transform

**Differentiation (review)**

\[ \mathcal{L}\{f'(t)\} = sF(s) - f(0) \]

\[ f(t) \quad \frac{d}{dt} \quad f'(t) \quad \frac{d}{dt} \quad f''(t) \]

\[ \mathcal{L}\{\} \quad \mathcal{L}\{\} \]

\[ F(s) \quad sF(s) - f(0) \]

\[ S \quad s\{sF(s) - f(0)\} - f'(0) \]

Example 1 (cont’d)

2. Partial fraction expansion

\[ Y(s) = \frac{-s^2 - s + 5}{s(s + 1)(s + 2)} = \frac{A}{s} + \frac{B}{s + 1} + \frac{C}{s + 2} \]

Multiply both sides by \( s \) & let \( s \) go to zero:

\[ sY(s)|_{s \to 0} = A + s \left|\frac{B}{s + 1}\right|_{s \to 0} + s \left|\frac{C}{s + 2}\right|_{s \to 0} \]

Similarly,

Example 1 (cont’d)

3. Inverse Laplace transform

\[ \mathcal{L}^{-1}\left\{ \frac{-s^2 - s + 5}{s(s + 1)(s + 2)} \right\} \]

If we are interested in only the final value of \( y(t) \), apply Final Value Theorem:

\[ \lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s) = \lim_{s \to 0} \frac{-s^2 - s + 5}{s + 1)(s + 2)} = \frac{5}{2} \]
Example 2

\[ \ddot{y}(t) - y(t) = t, \quad y(0) = 1, \dot{y}(0) = 1 \]

- S1
- S2
- S3 \[ y(t) = \mathcal{L}^{-1}(Y(s)) = \]

In this way, we can find a rather complicated solution to ODEs easily by using Laplace transform table!

Example: Newton’s law

\[ M \frac{d^2x(t)}{dt^2} = f(t) \]

We want to know the trajectory of \( x(t) \). By Laplace transform,

\[ M \left( s^2X(s) - sx(0) - x'(0) \right) = F(s) \]

\[ X(s) = \frac{1}{Ms^2} F(s) + \frac{x(0)}{s} + \frac{x'(0)}{s^2} \]

(Total response) = (Forced response) + (Initial condition response)

\[ x(t) = \mathcal{L}^{-1} \left[ \frac{1}{Ms^2} f'(s) \right] + x(0)u_x(t) + x'(0)t u_x(t) \]

Ex: Mechanical accelerometer
Ex: Mechanical accelerometer (cont’d)

- We would like to know how \( y(t) \) moves when unit step \( f(t) \) is applied with zero ICs.
- By Newton’s law
  \[
  \begin{align*}
  M \frac{d^2}{dt^2}(x(t) + y(t)) &= -b \frac{dy(t)}{dt} - ky(t) \\
  M_s \frac{d^2}{dt^2} x(t) &= f(t)
  \end{align*}
  \]

\[ \mathcal{L} Y(s) = \]

Ex: Mechanical accelerometer (cont’d)

- Suppose that \( b/M = 3 \), \( k/M = 2 \) and \( M_s = 1 \).
- Partial fraction expansion
  \[
  Y(s) = -\frac{1}{s^2 + 3s + 2} - \frac{1}{s} =
  \]
- Inverse Laplace transform
  \[
  y(t) =
  \]

Summary & Exercises

- Solution procedure to ODEs
  1. Laplace transform
  2. Partial fraction expansion
  3. Inverse Laplace transform

- Next, modeling of physical systems using Laplace transform

- Exercises
  - Derive the solution to the accelerometer problem.
  - E2.4 in the textbook.