ME451: Control Systems

Lecture 15
Time response of 2nd-order systems

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Course roadmap

Performance measures (review)

- Transient response
  - Peak value
  - Peak time
  - Percent overshoot
  - Delay time
  - Rise time
  - Settling time

- Steady state response
  - Steady state error

Next, we will connect these measures with s-domain.

Second-order systems

- A standard form of the second-order system

\[ G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \]

\[ \zeta : \text{damping ratio} \]
\[ \omega_n : \text{undamped natural frequency} \]

- DC motor position control example

\[ R(s) \quad \text{Amplifier} \quad E_a(s) \quad \text{Motor} \quad \Theta_m(s) \quad \text{Closed-loop TF} \]

\[ \Theta_m(s) = \frac{0.5}{s(s+2)} \]

\[ \frac{\Theta_m(s)}{R(s)} = \frac{0.5K}{s^2 + 2s + 0.5K} \]
Step response for 2nd-order system

- Input a unit step function to a 2nd-order system. What is the output?

\[
\begin{align*}
\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} & \quad u(t) \quad y(t) \\
0 & \quad 0 \\
1 & 
\end{align*}
\]

DC gain

\[G(0) = 1 \quad \lim_{t \to \infty} y(t) = G(0) = 1 \text{ if } G \text{ is stable}\]

Step response for 2nd-order system for various damping ratio

- Undamped
  \[\zeta = 0\]
- Underdamped
  \[0 < \zeta < 1\]
- Critically damped
  \[\zeta = 1\]
- Overdamped
  \[\zeta > 1\]

Step response for 2nd-order system

Underdamped case

- Math expression of \(y(t)\) for underdamped case

\[
Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{1}{s}
\]

\[
\mathcal{L}^{-1}\left[ y(t) \right] = 1 - \frac{e^{-\zeta \omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \cos^{-1}\zeta\right)
\]

Damped natural frequency

\[\omega_d = \omega_n \sqrt{1 - \zeta^2}\]

Peak value/time: Underdamped case

\[
y(t) = 1 - \frac{\zeta\pi}{\sqrt{1-\zeta^2}} e^{-\zeta \omega_n t} \sin\left(\omega_d t + \cos^{-1}\zeta\right)
\]

\[y(0) = 0 \quad y'(0) = 0\]

\[
t_{\text{max}} = \frac{\pi}{\omega_d}
\]

\[
\omega_d = \frac{2\pi}{2\pi}
\]

\[
\omega_d = \frac{2\pi}{2\pi}
\]
Properties of 2nd-order system

\[ G(s) = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} \quad 0 < \zeta < 1 \]

<table>
<thead>
<tr>
<th>Properties</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Peak time</td>
<td>( \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} )</td>
</tr>
<tr>
<td>Peak value</td>
<td>( 1 + e^{-\zeta \pi / \sqrt{1-\zeta^2}} )</td>
</tr>
<tr>
<td>Percent overshoot</td>
<td>( 100e^{-\zeta \pi / \sqrt{1-\zeta^2}} )</td>
</tr>
<tr>
<td>Settling time</td>
<td>( \approx \frac{3}{\zeta \omega_n} ) or ( \frac{4}{\zeta \omega_n} ) (5%) (2%)</td>
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</tbody>
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Some remarks

- Percent overshoot depends on \( \zeta \), but NOT \( \omega_n \).
- From 2nd-order transfer function, analytic expressions of delay & rise time are hard to obtain.
- Time constant is \( 1/(\zeta \omega_n) \), indicating convergence speed.
- For \( \zeta > 1 \), we cannot define peak time, peak value, percent overshoot.

P.O. vs. damping ratio

\[ 100e^{-\zeta \pi \sqrt{1-\zeta^2}} \]

Pole locations of \( G \)

- Poles (0<\( \zeta <1 \))

\[ s = -\zeta \omega_n \pm j\omega_n \sqrt{1-\zeta^2} \]

- Damping ratio

\[ \zeta = \cos \theta \]

Next, we clarify the influence of pole location on step response.
Influence of real part of poles

- Settling time $t_s$ decreases.

\[ y(t) = 1 - e^{-\zeta \omega_d t} \sin\left(\omega_d t + \cos^{-1} \zeta \right) \]

Influence of imag. part of poles

- Oscillation frequency $\omega_d$ increases.

\[ y(t) = 1 - \frac{e^{-\zeta \omega_d t}}{\sqrt{1 - \zeta^2}} \sin\left(\omega_d t + \cos^{-1} \zeta \right) \]

Influence of angle of poles

- Over/under-shoot decreases.

\[ y(t) = 1 - \frac{e^{-\zeta \omega_d t}}{\sqrt{1 - \zeta^2}} \sin\left(\omega_d t + \cos^{-1} \zeta \right) \]

\[ \zeta = \cos \theta \]

An example

- Require 5% settling time $t_s < t_{sm}$ (given):

\[ t_s \approx \frac{3}{\zeta \omega_n} < t_{sm} \rightarrow \zeta \omega_n > \frac{3}{t_{sm}} \]

\[ \omega_n \sqrt{1 - \zeta^2} (=: \omega_d) \]

\[ \frac{3}{t_{sm}} \]
An example (cont’d)

- Require PO < PO_m (given):
  \[ PO = 100e^{-\frac{\pi}{\tan \theta}} < PO_m \]
  \[ = 100e^{-\frac{\pi}{\tan \theta}} \]

\[ \begin{array}{|c|c|}
\hline
PO_m & \theta_m \\
\hline
4.3\% & \pi/4 \\
16.3\% & \pi/3 \\
\hline
\end{array} \]

\[ \zeta \omega_n > \frac{3}{t_{sm}} \]

Combination of two requirements

\[ \theta < \theta_m \]

Summary

- Transient response of 2nd-order system is characterized by:
  - Damping ratio \( \zeta \) & undamped natural frequency \( \omega_n \)
  - Pole locations
- Delay time and rise time are not so easy to characterize, and thus not covered in this course.
- For transient responses of high order systems, we need computer simulations.
- Next, Root locus

Exercises

(Use a calculator if necessary.)

- Read the related topics from the textbook.
- For the system below with \( \zeta=0.6, \omega_n=5 \) (rad/sec), obtain
  - Percent overshoot ?
  - 5% settling time ?

\[ \frac{1}{s} \]

\[ \frac{\omega_n^2}{s(s + 2\zeta \omega_n)} \]
Exercises

2. For the system below, design $K_1$ and $K_2$ s.t.
   - Percent overshoot is at most 20%?
   - Peak time is at most 1 sec.?
   - With designed $K_1$ and $K_2$, what is 5% settling time?

![Block diagram of a control system with unit step input and feedback connections]