Lecture 11
Routh-Hurwitz criterion: Control examples

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Stability summary (review)

Let $s_i$ be poles of rational $G$. Then, $G$ is ...

- (BIBO, asymptotically) stable if $\text{Re}(s_i) < 0$ for all $i$.
- marginally stable if
  - $\text{Re}(s_i) \leq 0$ for all $i$, and
  - simple root for $\text{Re}(s_i) = 0$
- unstable if
  it is neither stable nor marginally stable.

Routh-Hurwitz criterion (review)

$$Q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

The number of roots in the right half-plane is equal to the number of sign changes in the first column of Routh array.
Example 1

- Design $K(s)$ that stabilizes the closed-loop system for the following cases.
  - $K(s) = K$ (constant)
  - $K(s) = K_P + K_I/s$ (PI (Proportional-Integral) controller)

Example 1: $K(s) = K$

- Characteristic equation
  \[ 1 + K \frac{2}{s^3 + 4s^2 + 5s + 2} = 0 \]
  \[ s^3 + 4s^2 + 5s + 2 + 2K = 0 \]

- Routh array
  \[
  \begin{array}{c|ccc}
  s^3 & 1 & 5 \r & \\
  s^2 & 4 & 2 + 2K \r & \\
  s^1 & \frac{18 - 2K}{4} & 2 + 2K \r & \\
  s^0 & 2 + 2K & & \\
  \end{array}
  \]
  \[ -1 < K < 9 \]

Example 1: $K(s) = K_P + K_I/s$

- Characteristic equation
  \[ 1 + \left( K_P + \frac{K_I}{s} \right) \frac{2}{s^3 + 4s^2 + 5s + 2} = 0 \]
  \[ s^4 + 4s^3 + 5s^2 + (2 + 2K) s + 2K_I = 0 \]

- Routh array
  \[
  \begin{array}{c|ccc}
  s^4 & 1 & 2K_I \r & \\
  s^3 & 4 & 2 + 2K_P \r & \\
  s^2 & \frac{18 - 2K_P}{4} & 2K_I \r & \\
  s^1 & * & K_P < 9 \r & \\
  s^0 & 2K_I & K_I > 0 \r & \\
  \end{array}
  \]

Example 1: Range of $(K_P, K_I)$

- From Routh array,
  - $K_P < 9$
  - $K_I > 0$
  \[ (1 + K_P)(9 - K_P) - 8K_I > 0 \]
Example 1: \( K(s) = K_P + K_I/s \) (cont’d)
- Select \( K_P = 3 \) (<9)
- Routh array (cont’d)
  \[
  \begin{array}{cccc}
  s^4 & 1 & 5 & 2K_I \\
  s^3 & 4 & 8 & \\
  s^2 & 3 & 2K_I & \\
  s^1 & \frac{24 - 8K_I}{3} & 2K_I \\
  s^0 & 2K_I & \\
  \end{array}
  \]
  \[0 < K_I < 3\]
- If we select different \( K_P \), the range of \( K_I \) changes.

Example 1: What happens if \( K_P = K_I = 3 \)
- Auxiliary equation \( 3s^2 + 6 = 0 \) \( \Leftrightarrow s = \pm \sqrt{2}j \)
- Oscillation frequency \( \sqrt{2} \text{(rad/sec)} \)
- Period \( \frac{2\pi}{\sqrt{2}} \approx 4.4 \text{(sec)} \)

Example 2
- Determine the range of \( K \) and \( a \) that stabilize the closed-loop system.

Example 2 (cont’d)
Example 2 (cont’d)

- Characteristic equation
  \[
  1 + K \cdot \frac{1}{s(s+2)(s+3)} = 0
  \]
  \[
  1 + \frac{K}{s} \cdot \frac{1}{(s+2)(s+3)} = 0
  \]
  \[
  s(s+2)(s+3) + s + K = 0
  \]
  \[
  s^3 + 5s^2 + 7s + K = 0
  \]

- If K=35, oscillation frequency is obtained by the auxiliary equation
  \[
  5s^2 + 35 = 0 \iff s = \pm \sqrt{7}j
  \]

Example 2 (cont’d)

- Routh array
  \[
  \begin{array}{c|cc}
  s^3 & 1 & 7 \\
  s^2 & 5 & K \\
  s^1 & \frac{35-K}{5} & \\
  s^0 & K & \\
  \end{array}
  \]
  \[0 < K < 35\]

Summary and Exercises

- Control examples for Routh-Hurwitz criterion
  - P controller gain range for stability
  - PI controller gain range for stability
  - Oscillation frequency
  - Characteristic equation

- Next
  - Time domain specifications

- Exercises

More example 1

- \(Q(s) = s^3 + s^2 + s + 1 = (s+1)(s^2+1)\)

- Routh array
  \[
  \begin{array}{c|cc}
  s^3 & 1 & 1 \\
  s^2 & 1 & 1 \\
  s^1 & \sqrt{2} & \\
  s^0 & 1 & \\
  \end{array}
  \]

- Derivative of auxiliary poly.
  \[
  (s^2+1)' = 2s
  \]

- Auxiliary poly. is a factor of Q(s.)

- No sign changes in the first column
  \[\text{No root in OPEN(!) RHP}\]
More example 2

\[ Q(s) = s^5 + s^4 + 2s^3 + 2s^2 + s + 1 (= (s+1)(s^2+1)^2) \]

Routh array

\[
\begin{array}{c|ccc}
    s^5 & 1 & 2 & 1 \\
    s^4 & 1 & 2 & 1 \\
    s^3 & 4 & 0 & 4 \\
    s^2 & 1 & 1 & 0 \\
    s^1 & 2 & 0 & 0 \\
    s^0 & 1 & 0 & 0 \\
\end{array}
\]

Derivative of auxiliary poly.

\((s^4 + 2s^2 + 1)' = 4s^3 + 4s\)

\((s^2 + 1)' = 2s\)

No sign changes in the first column

No root in OPEN(!) RHP

More example 3

\[ Q(s) = s^4 - 1 (= (s + 1)(s - 1)(s^2 + 1)) \]

Routh array

\[
\begin{array}{c|ccc}
    s^4 & 1 & 0 & -1 \\
    s^3 & 4 & 0 & 0 \\
    s^2 & 4/\varepsilon & -1 & \varepsilon \varepsilon \\
    s^1 & 4/\varepsilon & -1 & 0 \\
    s^0 & -1 & 0 & 0 \\
\end{array}
\]

Derivative of auxiliary poly.

\((s^4 - 1)' = 4s^3\)

One sign changes in the first column

One root in OPEN(!) RHP