Stability summary (review)

Let $s_i$ be poles of rational $G$. Then, $G$ is …

- (BIBO, asymptotically) stable if $\text{Re}(s_i) < 0$ for all $i$.
- marginally stable if
  - $\text{Re}(s_i) \leq 0$ for all $i$, and
  - simple root for $\text{Re}(s_i) = 0$
- unstable if
  it is neither stable nor marginally stable.

Routh-Hurwitz criterion

- This is for LTI systems with a polynomial denominator (without sin, cos, exponential etc.)
- It determines if all the roots of a polynomial
  - lie in the open LHP (left half-plane),
  - or equivalently, have negative real parts.
- It also determines the number of roots of a polynomial in the open RHP (right half-plane).
- It does NOT explicitly compute the roots.
Polynomial and an assumption

- Consider a polynomial
  \[ Q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 \]
- Assume \( a_0 \neq 0 \)
  - If this assumption does not hold, \( Q \) can be factored as
    \[ Q(s) = s^m (\hat{a}_{n-m} s^{n-m} + \cdots + \hat{a}_1 s + \hat{a}_0) \]
    where \( \hat{a}_0 \neq 0 \)
  - The following method applies to the polynomial \( \hat{Q}(s) \)

Routh array

From the given polynomial

\[
\begin{array}{cccccc}
 s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
 s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\
 s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
 s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vphantom{\frac{a_n}{a_1}}
\end{array}
\]

(How to compute the third row)

\[
\begin{array}{cccccc}
 s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
 s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\
 s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
 s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vphantom{\frac{a_n}{a_1}}
\end{array}
\]

**Routh array (How to compute the third row)**

\[
b_1 = \frac{a_{n-2}a_{n-1} - a_{n}a_{n-3}}{a_{n-1}}
\]

\[
b_2 = \frac{a_{n-4}a_{n-1} - a_{n}a_{n-5}}{a_{n-1}}
\]

Routh array (How to compute the fourth row)

\[
\begin{array}{cccccc}
 s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
 s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\
 s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
 s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vphantom{\frac{a_n}{a_1}}
\end{array}
\]

**Routh array (How to compute the fourth row)**

\[
c_1 = \frac{a_{n-3}b_1 - a_{n-1}b_2}{b_1}
\]

\[
c_2 = \frac{a_{n-5}b_1 - a_{n-1}b_3}{b_1}
\]

\[
\vdots
\]
Routh-Hurwitz criterion

The number of roots in the open right half-plane is equal to the number of sign changes in the first column of the Routh array.

Example 1

\[ Q(s) = s^3 + s^2 + 2s + 8 = (s + 2)(s^2 - s + 4) \]

Routh array

\[
\begin{array}{c|cccc}
 s^3 & 1 & 2 & 2 - 8 \\
 s^2 & 1 & 8 & 1 \\
 s^1 & -6 & & \\
 s^0 & 8 & & \overbrace{8 \times (-6) - 0}^{-6} \\
\end{array}
\]

Two sign changes in the first column

\[ 1 \rightarrow -6 \rightarrow 8 \]

Two roots in RHP

\[ \frac{1}{2} \pm \frac{j\sqrt{15}}{2} \]

Example 2

\[ Q(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10 \]

Routh array

\[
\begin{array}{c|cc}
 s^5 & 1 & 2 & 11 \\
 s^4 & 2 & 4 & 10 \\
 s^3 & \varepsilon & 6 & \times \\
 s^2 & 4\varepsilon - 12 & \varepsilon & 10 \\
 s^1 & 6 & & \\
 s^0 & 10 & & \\
\end{array}
\]

If 0 appears in the first column of a nonzero row in the Routh array, replace it with a small positive number. In this case, Q has some roots in RHP.

Two sign changes in the first column

Two roots in RHP

\[ \varepsilon \rightarrow \frac{4\varepsilon - 12}{\varepsilon} \rightarrow 6 \]

Example 3

\[ Q(s) = s^4 + s^3 + 3s^2 + 2s + 2 \]

Routh array

\[
\begin{array}{c|ccc}
 s^4 & 1 & 3 & 2 \\
 s^3 & 1 & 2 & \times \\
 s^2 & 2 & 1 & 2 \\
 s^1 & 2 & & \\
 s^0 & 2 & & \times \\
\end{array}
\]

If zero row appears in the Routh array, Q has roots either on the imaginary axis or in RHP.

No sign changes in the first column

No roots in RHP

Take derivative of an auxiliary polynomial (which is a factor of Q(s))

\[ s^2 + 2 \]

But some roots are on the imaginary axis.
Example 4

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\[ Q(s) = s^3 + 3Ks^2 + (K + 2)s + 4 \]

Find the range of K s.t. Q(s) has all roots in the left half plane. (Here, K is a design parameter.)

Routh array

| \( s^3 \) | 1 \( K + 2 \) | \( s^2 \) | 3K | 4 | \( s^1 \) | \( \frac{3K(K+2)-4}{3K} \) | \( s^0 \) | 4 |
|---|---|---|---|---|---|---|---|

No sign changes in the first column

\[ \begin{cases} 3K > 0 \\ 3K(K + 2) - 4 > 0 \end{cases} \]

\[ K > -1 + \frac{\sqrt{21}}{3} \]

Example 4

Simple & important criteria for stability

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- **1st order polynomial** \( Q(s) = a_1s + a_0 \)
  All roots are in LHP \( \iff \) \( a_1 \) and \( a_0 \) have the same sign

- **2nd order polynomial** \( Q(s) = a_2s^2 + a_1s + a_0 \)
  All roots are in LHP \( \iff \) \( a_2, a_1 \) and \( a_0 \) have the same sign

- **Higher order polynomial** \( Q(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0 \)
  All roots are in LHP \( \iff \) All \( a_k \) have the same sign

Examples

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<table>
<thead>
<tr>
<th>( Q(s) )</th>
<th>All roots in open LHP?</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 3s + 5 )</td>
<td>Yes / No</td>
</tr>
<tr>
<td>( -2s^2 - 5s - 100 )</td>
<td>Yes / No</td>
</tr>
<tr>
<td>( 523s^2 - 57s + 189 )</td>
<td>Yes / No</td>
</tr>
<tr>
<td>( (s^2 + s - 1)(s^2 + s + 1) )</td>
<td>Yes / No</td>
</tr>
<tr>
<td>( s^3 + 5s^2 + 10s - 3 )</td>
<td>Yes / No</td>
</tr>
</tbody>
</table>

Summary and Exercises

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- Routh-Hurwitz stability criterion
  - Routh array
  - Routh-Hurwitz criterion is applicable to only polynomials (so, it is not possible to deal with exponential, sin, cos etc.).

- Next,
  - Routh-Hurwitz criterion in control examples

- Exercises
  - Read Routh-Hurwitz criterion in the textbook.
  - Do Examples.