Please start with an easy question and try to answer all questions.

Problem:	1	2	Total
Max. Grade:	30	80	100+10(extra)
Grade:	30	80	110

great!

1 Problem

Consider the second order system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

and the following two performance specifications:

1. $G(s)$ is stable,
2. Percent Overshoot (PO) $> 4.3\%$, and
3. 5% settling time < 3 sec.

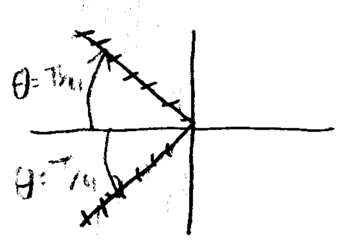
Provide answers to the following questions.

a. (10 points) Find (or draw) all possible locations of the two poles of $G(s)$ in the complex domain to satisfy stability and PO conditions.

b. (10 points) Find all possible locations of the two poles of $G(s)$ in the complex domain to satisfy the settling time condition.

a) $PO > 4.3\%$

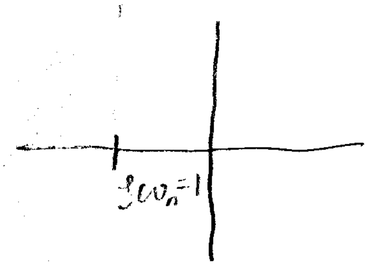
$\Rightarrow \theta > \theta_r = \pi/4$



b) 5% settling time < 3sec

$t_s < t_{sm}$

$\zeta\omega_n > \frac{3}{t_{sm}} = \frac{3}{3} = 1$



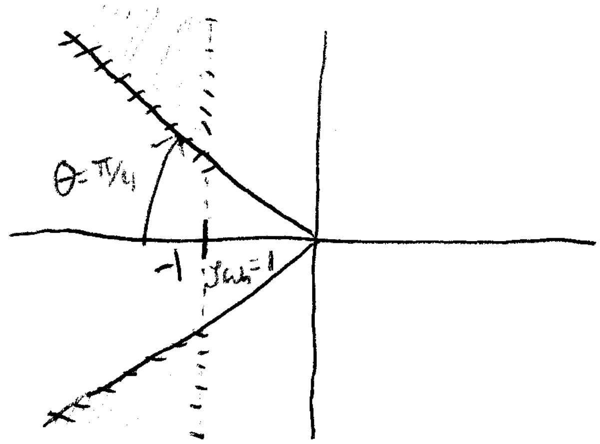
c. (10 points) Find all possible locations of the two poles of $G(s)$ in the complex domain to satisfy stability, PO and settling time conditions.

Hint:

$$PO = 100e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} = 100e^{-\pi/\tan\theta},$$

where $\zeta = \cos\theta$. From this equation, we know that $\theta = \pi/4$ when $PO = 4.3\%$.

c) $PO > 4.3\% \Rightarrow \theta > \theta_m = \pi/4$



5% settling time

$t_s < 3$ sec

$\Rightarrow \zeta\omega_n > \frac{3}{t_{sm}} = \frac{3}{3}$

2 Problem

Consider the transfer function

$$G(s) = \frac{s+1}{(s^2+2^2)(s+3)}$$

in the feedback system in Fig. 1.

a. (5 points) What is the root locus? Explain by using Fig. 1.

b. (15 points) Draw the root locus by varying k from 0 to ∞. What is the relative degree? Show the intersection of asymptotes.

c. (10 points) Find the range of $k > 0$ for which the closed-loop system is stable.

d. (10 extra points) Compute the angle of departures for the complex poles of the open-loop transfer function.

e. (15 points) For a unit step input $U(S) = \frac{1}{s}$, compute the steady state error $e_{ss} := \lim_{t\to\infty}[e(t) := u(t) - y(t)]$ in terms of k.

f. (15 points) Propose a dynamical controller (in stead of using a constant gain k) in order to eliminate the steady state error for the unit step input completely. Justify your answer.

g. (10 points) Explain how you prove the stability of the closed-loop system with the dynamical controller you provided.

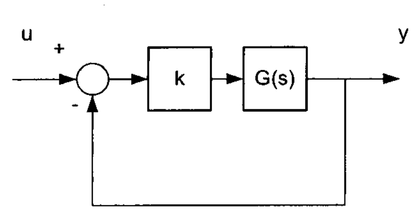
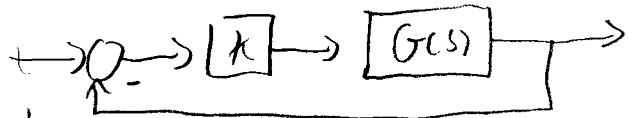


Figure 1: A closed-loop system.

a) The root locus is a visual representation of the possible pole locations in Figure 1, assuming the constant k is in the range $0 < k < \infty$. This can also tell you the natural frequency, damping ratio, and damping frequency of the system for a given k as well as the stability.

→

b)

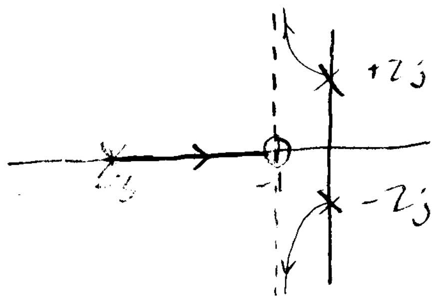


$$G(s) = \frac{s+1}{(s^2+2^2)(s+3)}$$

Assume stable CL transfer function (poles in left half plane)

Zeros: -1

poles: $\pm 2j$, -3



relative degree

r = degree(den) - degree(num)

$= 3 - 1 = 2$

$$\text{Intersection} = \frac{\sum \text{Poles} - \sum \text{Zeros}}{r}$$

$$= \frac{(+2j-2j-3)-(-1)}{2}$$

$$= \frac{-2}{2} = -1$$

c) Characteristic eq of closed loop

$$(s^2+4)(s+3) + K(s+1) = 0$$

$$s^3 + 3s^2 + (4+K)s + 12+K = 0$$

Routh Array

s^3	1	$4+k$
s^2	3	$12+k$
s^1	$\frac{2}{3}k$	
s^0	$12+k$	

$$\frac{3(4+k)-(12+k)}{3}$$

$$= 4+k-4-\frac{k}{3}$$

$$= \frac{2}{3}k$$

For a stable system

$-12 < K$

$+\ 0 < K$

$-12 < 0 \quad \therefore \quad \boxed{K > 0}$

$\longrightarrow$

d) $L(s) = \frac{s+1}{(s^2+4)(s+3)} = \frac{s+1}{(s+2j)(s-2j)(s+3)}$

$$\lim_{s_0 \to P_1} \angle L(s) = \angle(s_0 - Z_1) - \underbrace{\angle(s_0 - P_1)}_{\theta_{dep}} - \angle(s_0 - P_2) - \angle(s_0 - P_3) = \pm 180$$

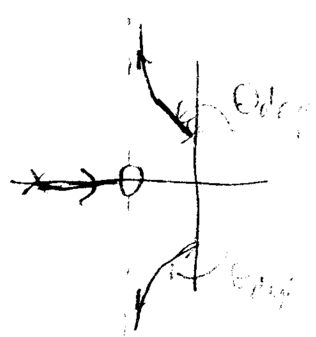
$$\angle(s_0 - P_1) = \angle(P_1 - Z_1) - \angle(P_1 - P_2) - \angle(P_1 - P_3) + 180$$

$$P_1 - Z_1 = 2j - (-1) = 1 + 2j$$
$$\angle(P_1 - Z_1) = \tan^{-1}(2/1) = 63.43°$$

$$P_1 - P_2 = 2j - (-2j) = 4j$$
$$\angle(P_1 - P_2) = 90°$$

$$P_1 - P_3 = 2j - (-3) = 3 + 2j$$
$$\angle(P_1 - P_3) = \tan^{-1}(2/3) = 33.69°$$

$$\theta_{dep} = 63.43° - 90° - 33.69° + 180°$$
$$= 119.74°$$



→

e) $u(t)$ is a unit step

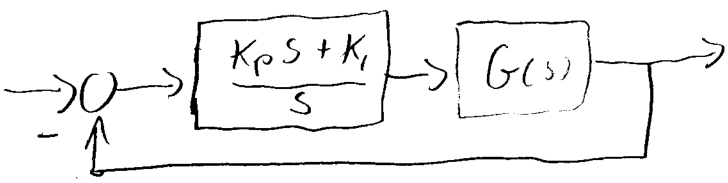
$U(s) = \frac{1}{s}$

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+K/12}$$

$$K_p = \lim_{s \to 0} G(s) = K\left(\frac{0+1}{(0^2+4)(0+3)}\right) = \frac{K}{12}$$

f) I would suggest adding a PI Controller. This would increase the system's type from a 0 to a 1.

$$PI = \frac{K_p s + K_I}{s}$$



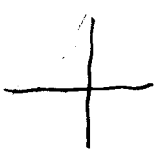
open loop transfer function becomes $\frac{(\ \ )}{s(\ \ )}$

$$e_{ss} = \frac{1}{1+K_p} \; ; \text{ where } K_p = \lim_{s \to 0} G(s) = \lim_{s \to 0} \frac{(\ )}{s(\ )} = \infty$$

$$e_{ss} = \frac{1}{1+\infty} = 0$$

g) The stability can be proved either by substituting the new characteristic eq into the Routh Array or by drawing the root locus.

If the root locus is only in the left half plane the system is stable.



If the first column of the Routh array doesn't change signs the system is stable.

s^4	( ) . . .
s^3	( ) . . .
s^2	( ) . . .
s^1	( ) . . .
s^0	( ) . . .