

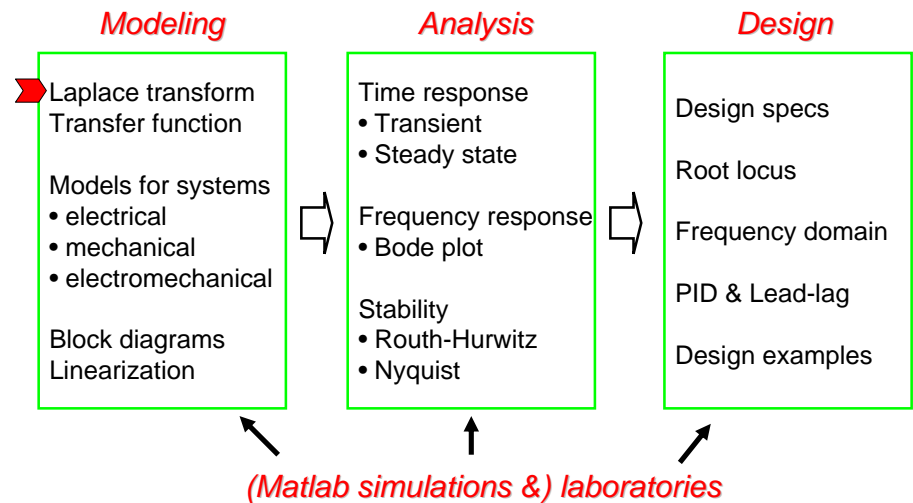
ME451: Control Systems

Lecture 3

Solution to ODEs via Laplace transform

Dr. Jongeun Choi
 Department of Mechanical Engineering
 Michigan State University

Course roadmap

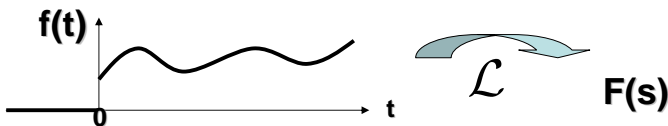


Laplace transform (review)

- One of most important math tools in the course!
- Definition: For a function $f(t)$ ($f(t)=0$ for $t<0$),

$$F(s) = \mathcal{L}\{f(t)\} := \int_0^{\infty} f(t)e^{-st} dt$$

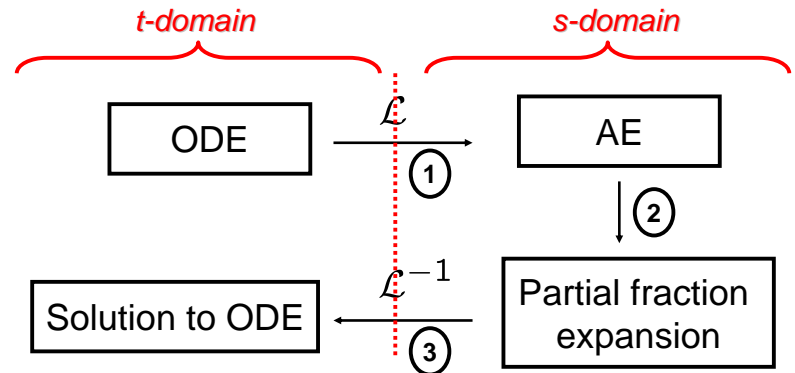
(s: complex variable)



- We denote Laplace transform of $f(t)$ by $F(s)$.

An advantage of Laplace transform

- We can transform an ordinary differential equation (ODE) into an algebraic equation (AE).



Example 1

ODE with initial conditions (ICs)

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = 5u_s(t), \quad y(0) = -1, \quad y'(0) = 2$$

1. Laplace transform

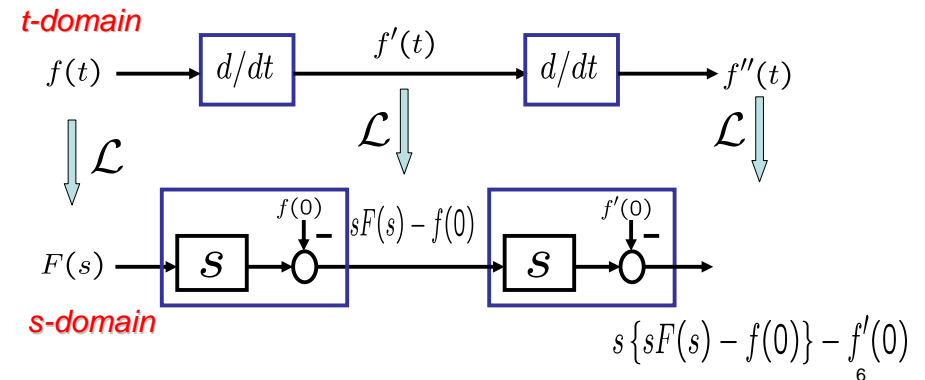
$$\underbrace{s^2Y(s) - sy(0) - y'(0)}_{\mathcal{L}\{y''(t)\}} + 3 \underbrace{\{sY(s) - y(0)\}}_{\mathcal{L}\{y'(t)\}} + 2Y(s) = \frac{5}{s}$$

$$\Rightarrow Y(s) = \frac{-s^2 - s + 5}{s(s+1)(s+2)}$$

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Properties of Laplace transform Differentiation (review)

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$



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Example 1 (cont'd)

2. Partial fraction expansion

$$Y(s) = \frac{-s^2 - s + 5}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

unknowns

Multiply both sides by s & let s go to zero:

$$sY(s)|_{s \rightarrow 0} = A + s \frac{B}{s+1} \Big|_{s \rightarrow 0} + s \frac{C}{s+2} \Big|_{s \rightarrow 0} \Rightarrow A = sY(s)|_{s \rightarrow 0} = \frac{5}{2}$$

Similarly,

$$B = (s+1)Y(s)|_{s \rightarrow -1} = \dots = -5$$

$$C = (s+2)Y(s)|_{s \rightarrow -2} = \dots = \frac{3}{2}$$

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Example 1 (cont'd)

3. Inverse Laplace transform

$$\mathcal{L}^{-1}\left\{Y(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}\right\}$$

$$\Rightarrow y(t) = \left(\frac{5}{2} + \underbrace{(-5)}_B e^{-t} + \frac{3}{2} e^{-2t} \right) u_s(t)$$

If we are interested in only the final value of y(t), apply Final Value Theorem:

$$\lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} sY(s) = \lim_{s \rightarrow 0} \frac{-s^2 - s + 5}{(s+1)(s+2)} = \frac{5}{2}$$

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Example 2

$$\ddot{y}(t) - y(t) = t, \quad y(0) = 1, \dot{y}(0) = 1$$

▪ S1 $s^2 Y(s) - sy(0) - \dot{y}(0) - Y(s) = \frac{1}{s^2},$

▪ S2
$$Y(s) = \frac{1}{s-1} + \frac{1}{s^2(s^2-1)}$$

$$= \frac{1}{s-1} + \frac{s^2 - (s^2-1)}{s^2(s^2-1)} = \frac{1}{s-1} + \frac{1}{s^2-1} - \frac{1}{s^2}$$

$$= \frac{1}{s-1} + \frac{1}{2s-1} - \frac{1}{2s+1} - \frac{1}{s^2}$$

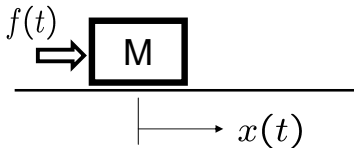
▪ S3 $y(t) = \mathcal{L}^{-1}(Y(s)) = \left[\frac{3}{2}e^t - \frac{1}{2}e^{-t} - t \right] u_s(t)$

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In this way, we can find a rather complicated solution to ODEs easily by using Laplace transform table!

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Example: Newton's law

$$M \frac{d^2 x(t)}{dt^2} = f(t)$$


We want to know the trajectory of $x(t)$. By Laplace transform,

$$M(s^2 X(s) - sx(0) - x'(0)) = F(s)$$

$$\Rightarrow X(s) = \frac{1}{Ms^2} F(s) + \frac{x(0)}{s} + \frac{x'(0)}{s^2}$$

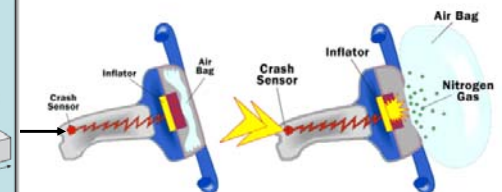
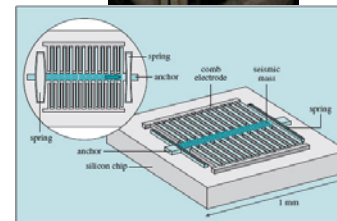
(Total response) = (Forced response) + (Initial condition response)

$$\Rightarrow x(t) = \mathcal{L}^{-1} \left[\frac{1}{Ms^2} F(s) \right] + x(0)u_s(t) + x'(0)tu_s(t)$$

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EX. Air bag and accelerometer

- Tiny MEMS accelerometer
 - Microelectromechanical systems (MEMS)

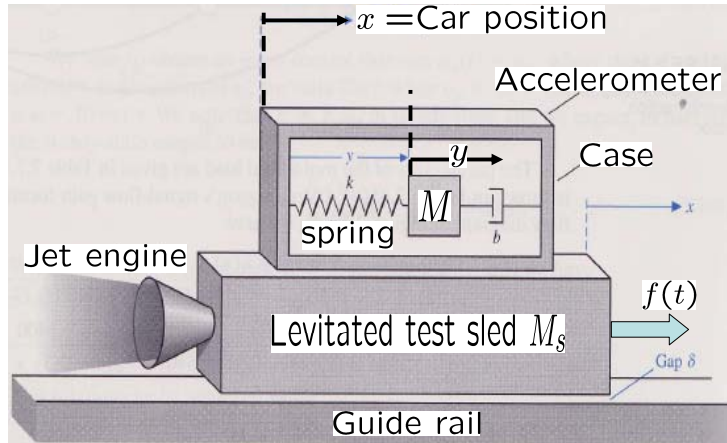


(Pictures from various websites)

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Ex: Mechanical accelerometer

- Taken from Dorf & Bishop book



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Ex: Mechanical accelerometer (cont'd)

- We would like to know how $y(t)$ moves when unit step $f(t)$ is applied with zero ICs.

- By Newton's law

$$\begin{cases} M \frac{d^2}{dt^2}(x(t) + y(t)) = -b \frac{dy(t)}{dt} - ky(t) \\ M_s \frac{d^2 x(t)}{dt^2} = f(t) \end{cases}$$

$$\rightarrow My''(t) + by'(t) + ky(t) = -\frac{M}{M_s} f(t)$$

$$\mathcal{L} \rightarrow Y(s) = -\frac{M}{M_s} \cdot \frac{1}{Ms^2 + bs + k} \cdot \frac{1}{s} = -\frac{1}{M_s} \cdot \frac{1}{s^2 + (b/M)s + (k/M)} \cdot \frac{1}{s}$$

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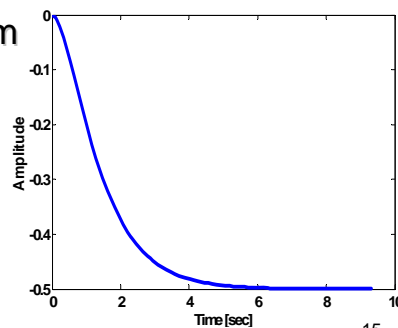
Ex: Mechanical accelerometer (cont'd)

- Suppose that $b/M=3$, $k/M=2$ and $M_s=1$.
- Partial fraction expansion

$$Y(s) = -\frac{1}{s^2 + 3s + 2} \cdot \frac{1}{s} = -\frac{1}{2s} + \frac{1}{s+1} - \frac{1}{2(s+2)}$$

- Inverse Laplace transform

$$y(t) = \left(-\frac{1}{2} + e^{-t} - \frac{1}{2}e^{-2t} \right) u_s(t)$$



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Summary & Exercises

- Solution procedure to ODEs
 1. Laplace transform
 2. Partial fraction expansion
 3. Inverse Laplace transform
- Next, modeling of physical systems using Laplace transform
- Exercises
 - Read Appendix B.
 - Derive the solution to the accelerometer problem.
 - Solve Problems:
 - B.3-(a), (b) in page 648
 - B.10 in page 649.

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