ME451: Control Systems

Lecture 2
Laplace transform

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Course roadmap

Modeling	*Analysis*	*Design*
Laplace transform Transfer function Models for systems • electrical • mechanical • electromechanical Block diagrams Linearization	Time response • Transient • Steady state Frequency response • Bode plot Stability • Routh-Hurwitz • (Nyquist)	Design specs Root locus Frequency domain PID & Lead-lag Design examples

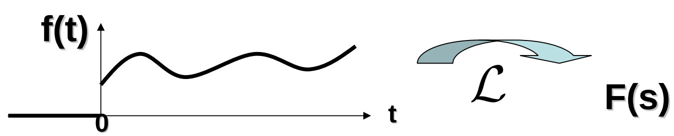
(Matlab simulations &) laboratories

Laplace transform

- One of most important math tools in the course!
- Definition: For a function f(t) (f(t)=0 for t<0),

$$F(s) = \mathcal{L}\{f(t)\} := \int_0^\infty f(t)e^{-st}dt$$

(s: complex variable)

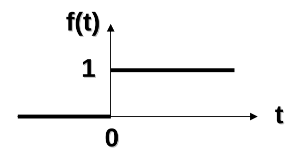


- We denote Laplace transform of f(t) by F(s).

Examples of Laplace transform

- Unit step function

$$f(t) = u_s(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

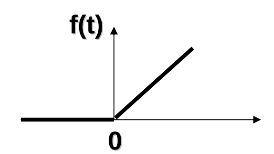


$$\Rightarrow F(s) = \int_0^\infty 1 \cdot e^{-st}dt = -\frac{1}{s}\left[e^{-st}\right]_0^\infty = \frac{1}{s}$$

(Memorize this!)

- Unit ramp function

$$f(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$



$$\Rightarrow F(s) = \int_0^\infty te^{-st}dt = -\frac{1}{s}\left[te^{-st}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}dt = \frac{1}{s^2}$$

(Integration by parts)

Integration by parts

$$\frac{d(u(t)v(t))}{dt} =: (u(t)v(t))' = u'(t)v(t) + u(t)v'(t)$$

$$\int (uv)'dt = \int u'vdt + \int uv'dt$$

$$uv = \int u'vdt + \int uv'dt$$

$$\int_a^b uv'dt = [uv]_a^b - \int_a^b u'vdt$$

EX. $\int_0^\infty te^{-st}dt?$ Let $u(t) = t,\ v'(t) = e^{-st}$ $\quad u'(t) = 1,\ v(t) = -\frac{1}{s}e^{-st}$

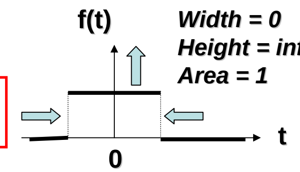
$$\int_0^\infty te^{-st}dt = \left[t\left(-\frac{1}{s}e^{-st}\right)\right]_0^\infty - \int_0^\infty 1\left(-\frac{1}{s}e^{-st}\right)dt = \frac{1}{s^2}$$

Examples of Laplace transform (cont'd)

- Unit impulse function

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

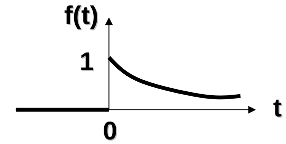
$$\int_{-\infty}^{\infty} \delta(t)g(t)dt = g(0)$$



$$\Rightarrow F(s) = \int_0^\infty \delta(t)e^{-st}dt = e^{-s\cdot 0} = 1$$ ***(Memorize this!)***

- Exponential function

$$f(t) = \begin{cases} e^{-\alpha t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$



$$\Rightarrow F(s) = \int_0^\infty e^{-\alpha t}\cdot e^{-st}dt = -\frac{1}{s+\alpha}\left[e^{-(s+\alpha)t}\right]_0^\infty = \frac{1}{s+\alpha}$$

Examples of Laplace transform (cont'd)

- Sine function $\mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2+\omega^2}$
- Cosine function $\mathcal{L}\{\cos \omega t\} = \frac{s}{s^2+\omega^2}$

(Memorize these!)

Remark: Instead of computing Laplace transform for each function, and/or memorizing complicated Laplace transform, use the ***Laplace transform table*** !

Laplace transform table
(Table B.1 in Appedix B of the textbook)

	Time Function $f(t) = \mathcal{L}^{-1}\{F(s)\}$	Laplace Transform $F(s) = \mathcal{L}\{f(t)\}$
1	1	$\frac{1}{s}$ $\quad s > 0$
2	t (unit-ramp function)	$\frac{1}{s^2}$ $\quad s > 0$
3	t^n (n, a positive integer)	$\frac{n!}{s^{n+1}}$ $\quad s > 0$
4	e^{at}	$\frac{1}{s-a}$ $\quad s > a$
5	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$ $\quad s > 0$
6	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$ $\quad s > 0$
7	$t^n g(t)$, for $n = 1, 2, \ldots$	$(-1)^n \frac{d^n G(s)}{ds^n}$
8	$t \sin \omega t$	$\frac{2\omega s}{(s^2+\omega^2)^2}$ $\quad s > \|\omega\|$
9	$t \cos \omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$ $\quad s > \|\omega\|$
10	$g(at)$	$\frac{1}{a}G\left(\frac{s}{a}\right)$ Scale property
11	$e^{at}g(t)$	$G(s-a)$ Shift property
12	$e^{at}t^n$, for $n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}$ $\quad s > a$

$\mathcal{L}$ → ; ← $\mathcal{L}^{-1}$

Inverse Laplace Transform

Properties of Laplace transform
1. Linearity

$$\mathcal{L}\{\alpha_1 f_1(t) + \alpha_2 f_2(t)\} = \alpha_1 F_1(s) + \alpha_2 F_2(s)$$

Proof.

$$\begin{aligned}\mathcal{L}\{\alpha_1 f_1(t) + \alpha_2 f_2(t)\} &= \int_0^\infty (\alpha_1 f_1(t) + \alpha_2 f_2(t)) e^{-st} dt \\ &= \alpha_1 \underbrace{\int_0^\infty f_1(t) e^{-st} dt}_{F_1(s)} + \alpha_2 \underbrace{\int_0^\infty f_2(t) e^{-st} dt}_{F_2(s)}\end{aligned}$$

Ex.

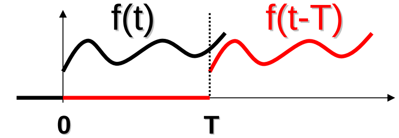
$$\mathcal{L}\left\{5u_s(t) + 3e^{-2t}\right\} = 5\mathcal{L}\{u_s(t)\} + 3\mathcal{L}\left\{e^{-2t}\right\} = \frac{5}{s} + \frac{3}{s+2}$$

Properties of Laplace transform
2. Time delay

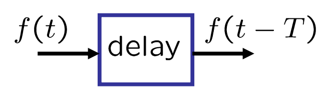
$$\mathcal{L}\{f(t-T)u_s(t-T)\} = e^{-Ts}F(s)$$

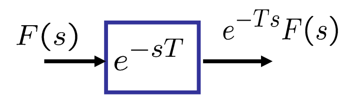
Proof.

$$\begin{aligned}&\mathcal{L}\{f(t-T)u_s(t-T)\} \\ &= \int_T^\infty f(t-T)e^{-st}dt \\ &= \int_0^\infty f(\tau)e^{-s(T+\tau)}d\tau = e^{-sT}F(s)\end{aligned}$$



Ex. $\mathcal{L}\left\{e^{-0.5(t-4)}u_s(t-4)\right\} = \dfrac{e^{-4s}}{s+0.5}$





 t-domain *s-domain*

Properties of Laplace transform
3. Differentiation

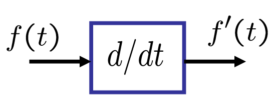
$$\mathcal{L}\left\{f'(t)\right\} = sF(s) - f(0)$$

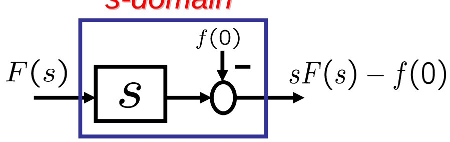
Proof.

$$\mathcal{L}\left\{f'(t)\right\} = \int_0^\infty f'(t)e^{-st}dt = \left[f(t)e^{-st}\right]_0^\infty + s\int_0^\infty f(t)e^{-st}dt = sF(s) - f(0)$$

Ex. $\mathcal{L}\left\{(\cos 2t)'\right\} = s\mathcal{L}\{\cos 2t\} - 1 = \dfrac{s^2}{s^2+4} - 1 = \dfrac{-4}{s^2+4} (= \mathcal{L}\{-2\sin 2t\})$

t-domain *s-domain*





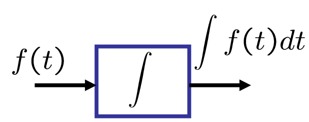
Properties of Laplace transform
4. Integration

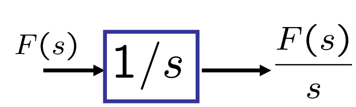
$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$$

Proof.

$$\begin{aligned}\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] &= \int_0^\infty \left(\int_0^t f(\tau)d\tau\right)e^{-st}dt \\ &= -\frac{1}{s}\left[\left(\int_0^t f(\tau)d\tau\right)e^{-st}\right]_0^\infty + \frac{1}{s}\int_0^\infty f(t)e^{-st}dt = \frac{F(s)}{s}\end{aligned}$$

t-domain *s-domain*





Properties of Laplace transform
5. Final value theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$ if all the poles of sF(s) are in the left half plane (LHP)

Ex. $F(s) = \dfrac{5}{s(s^2+s+2)} \Rightarrow \lim_{t\to\infty} f(t) = \lim_{s\to 0} \dfrac{5}{s^2+s+2} = \dfrac{5}{2}$

Poles of sF(s) are in LHP, so final value thm applies.

Ex. $F(s) = \dfrac{4}{s^2+4} \Rightarrow \lim_{t\to\infty} f(t) \neq \lim_{s\to 0} \dfrac{4s}{s^2+4} = 0$

Some poles of sF(s) are not in LHP, so final value thm does NOT apply.

Properties of Laplace transform
6. Initial value theorem

$$\lim_{t\to 0+} f(t) = \lim_{s\to\infty} sF(s)$$ if the limits exist.

Remark: In this theorem, it does not matter if pole location is in LHS or not.

Ex. $F(s) = \dfrac{5}{s(s^2+s+2)} \Rightarrow \lim_{t\to 0+} f(t) = \lim_{s\to\infty} sF(s) = 0$

Ex. $F(s) = \dfrac{4}{s^2+4} \Rightarrow \lim_{t\to 0+} f(t) = \lim_{s\to\infty} sF(s) = 0$

Properties of Laplace transform
7. Convolution

$$\left.\begin{array}{l} F_1(s) = \mathcal{L}\{f_1(t)\} \\ F_2(s) = \mathcal{L}\{f_2(t)\} \end{array}\right\}$$

Convolution

$$\Rightarrow F_1(s)F_2(s) = \mathcal{L}\left\{\overbrace{\int_0^t f_1(\tau) f_2(t-\tau)\,d\tau}\right\} = \mathcal{L}\left\{\int_0^t f_1(t-\tau) f_2(\tau)\,d\tau\right\}$$

IMPORTANT REMARK

$$F_1(s)F_2(s) \neq \mathcal{L}\{f_1(t) f_2(t)\}$$

Summary & Exercises

- Laplace transform (Important math tool!)
  - Definition
  - Laplace transform table
  - Properties of Laplace transform
- Next
  - Solution to ODEs via Laplace transform
- Exercises
  - Read Appendix A, B.
  - Solve Problems B.1 (a), (b); B.2 (a), (c), (d).