ME451: Control Systems

Lecture 2
Laplace transform

Prof. Clark Radcliffe, Prof. Jongeun Choi
Department of Mechanical Engineering
Michigan State University

Course roadmap

Modeling

- Laplace transform
- Transfer function
- Models for systems
  - electrical
  - mechanical
  - electromechanical
- Block diagrams
- Linearization

Analysis

- Time response
  - Transient
  - Steady state
- Frequency response
  - Bode plot
- Stability
  - Routh-Hurwitz
  - (Nyquist)

Design

- Design specs
- Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

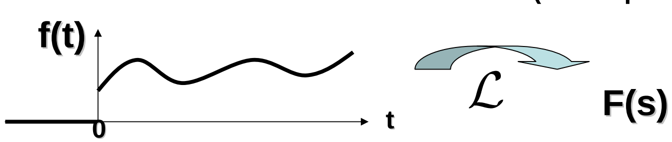
(Matlab simulations &) laboratories

Laplace transform

- One of most important math tools in the course!
- Definition: For a function f(t) (f(t)=0 for t<0),

$$F(s) = \mathcal{L}\{f(t)\} := \int_0^\infty f(t)e^{-st}dt$$

(s: complex variable)

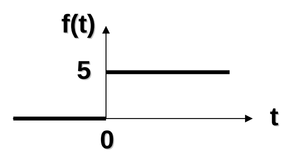


- We denote Laplace transform of f(t) by F(s).

Example of Laplace transform

- Step function

$$f(t) = 5u(t) = \begin{cases} 5 & t \geq 0 \\ 0 & t < 0 \end{cases}$$



$$F(s) = \int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty 5e^{-st}\,dt = 5\int_0^\infty e^{-st}\,dt$$

$$= 5\left[-\frac{1}{s}\left[e^{-st}\right]_0^\infty\right] = 5\left[\frac{1}{s}\right] = \frac{5}{s}$$

Remember L(u(t)) = 1/s

Integration is Hard

Tables are Easier

Laplace transform table
(Table B.1 in Appendix B of the textbook)

	Time Function $f(t) = \mathcal{L}^{-1}\{F(s)\}$	Laplace Transform $F(s) = \mathcal{L}\{f(t)\}$
1	1	$\frac{1}{s}$ $s > 0$
2	t (unit-ramp function)	$\frac{1}{s^2}$ $s > 0$
3	t^n (n, a positive integer)	$\frac{n!}{s^{n+1}}$ $s > 0$
4	e^{at}	$\frac{1}{s-a}$ $s > a$
5	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$ $s > 0$
6	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$ $s > 0$
7	$t^n g(t)$, for $n = 1, 2, \ldots$	$(-1)^n \frac{d^n G(s)}{ds^n}$
8	$t \sin \omega t$	$\frac{2\omega s}{(s^2+\omega^2)^2}$ $s > \lvert\omega\rvert$
9	$t \cos \omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$ $s > \lvert\omega\rvert$
10	$g(at)$	$\frac{1}{a}G\left(\frac{s}{a}\right)$ Scale property
11	$e^{at}g(t)$	$G(s-a)$ Shift property
12	$e^{at}t^n$, for $n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}$ $s > a$

$\mathcal{L}$ →

← $\mathcal{L}^{-1}$

Inverse Laplace Transform

Properties of Laplace transform
Linearity

$$\mathcal{L}\{\alpha_1 f_1(t) + \alpha_2 f_2(t)\} = \alpha_1 F_1(s) + \alpha_2 F_2(s)$$

Proof.

$$\begin{aligned}\mathcal{L}\{\alpha_1 f_1(t) + \alpha_2 f_2(t)\} &= \int_0^\infty (\alpha_1 f_1(t) + \alpha_2 f_2(t)) e^{-st} dt \\ &= \alpha_1 \underbrace{\int_0^\infty f_1(t) e^{-st} dt}_{F_1(s)} + \alpha_2 \underbrace{\int_0^\infty f_2(t) e^{-st} dt}_{F_2(s)}\end{aligned}$$

Ex.

$$\mathcal{L}\left\{5u_s(t) + 3e^{-2t}\right\} = 5\mathcal{L}\{u_s(t)\} + 3\mathcal{L}\left\{e^{-2t}\right\} = \frac{5}{s} + \frac{3}{s+2}$$

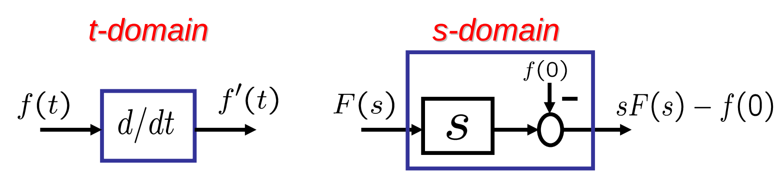
Properties of Laplace transform
Differentiation

$$\mathcal{L}\left\{f'(t)\right\} = sF(s) - f(0)$$

Proof.

$$\mathcal{L}\left\{f'(t)\right\} = \int_0^\infty f'(t) e^{-st} dt = \left[f(t)e^{-st}\right]_0^\infty + s\int_0^\infty f(t) e^{-st} dt = sF(s) - f(0)$$

Ex. $\mathcal{L}\left\{(\cos 2t)'\right\} = s\mathcal{L}\{\cos 2t\} - 1 = \frac{s^2}{s^2+4} - 1 = \frac{-4}{s^2+4} (= \mathcal{L}\{-2\sin 2t\})$



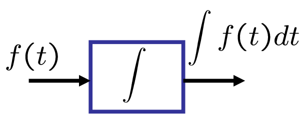
Properties of Laplace transform

Integration

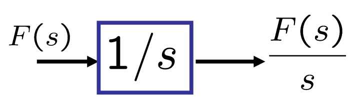
$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$$

Proof.
$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \int_0^\infty \left(\int_0^t f(\tau)d\tau\right)e^{-st}dt$$
$$= -\frac{1}{s}\left[\left(\int_0^t f(\tau)d\tau\right)e^{-st}\right]_0^\infty + \frac{1}{s}\int_0^\infty f(t)e^{-st}dt = \frac{F(s)}{s}$$

t-domain



s-domain



Properties of Laplace transform

Final value theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$ if all the poles of sF(s) are in the left half plane (LHP)

Ex. $F(s) = \dfrac{5}{s(s^2+s+2)} \Rightarrow \lim_{t\to\infty} f(t) = \lim_{s\to 0}\dfrac{5}{s^2+s+2} = \dfrac{5}{2}$

Poles of sF(s) are in LHP, so final value thm applies.

Ex. $F(s) = \dfrac{4}{s^2+4} \Rightarrow \lim_{t\to\infty} f(t) \neq \lim_{s\to 0}\dfrac{4s}{s^2+4} = 0$

Some poles of sF(s) are not in LHP, so final value thm does NOT apply.

Properties of Laplace transform

Initial value theorem

$$\lim_{t\to 0+} f(t) = \lim_{s\to\infty} sF(s)$$ if the limits exist.

Remark: In this theorem, it does not matter if pole location is in LHP or not.

Ex. $F(s) = \dfrac{5}{s(s^2+s+2)} \Rightarrow \lim_{t\to 0+} f(t) = \lim_{s\to\infty} sF(s) = 0$

Ex. $F(s) = \dfrac{4}{s^2+4} \Rightarrow \lim_{t\to 0+} f(t) = \lim_{s\to\infty} sF(s) = 0$

Properties of Laplace transform

Convolution

$$\left.\begin{array}{l} F_1(s) = \mathcal{L}\{f_1(t)\} \\ F_2(s) = \mathcal{L}\{f_2(t)\} \end{array}\right\}$$

Convolution

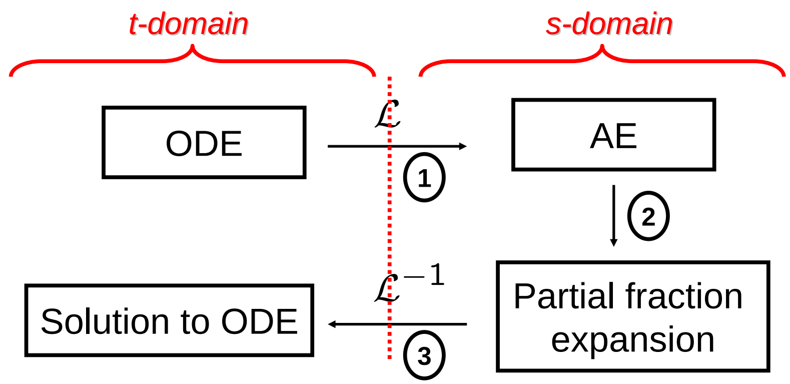
$$\Rightarrow \quad F_1(s)F_2(s) = \mathcal{L}\left\{\overbrace{\int_0^t f_1(\tau)f_2(t-\tau)d\tau}\right\}$$
$$= \mathcal{L}\left\{\int_0^t f_1(t-\tau)f_2(\tau)d\tau\right\}$$

IMPORTANT REMARK

$$F_1(s)F_2(s) \neq \mathcal{L}\{f_1(t)f_2(t)\}$$
$$L^{-1}\left(F_1(s)F_2(s)\right) \neq f_1(t)f_2(t)$$

An advantage of Laplace transform

- We can transform an ordinary differential equation (ODE) into an algebraic equation (AE).



Example 1

- 1st Order ODE with input and Initial Condition

$$5\dot{y}(t)+10y(t)=3u(t) \qquad y(0)=1$$

- Take Laplace Transform

$$5\left[sY(s)-y(0)\right]+10\left[Y(s)\right]=3\left[\frac{1}{s}\right]$$

- Solve for $Y(s)$

$$(5s+10)Y(s)=5y(0)+3\left[\frac{1}{s}\right]$$

$$Y(s)=\frac{5}{(5s+10)}+\frac{3}{s(5s+10)}=\frac{1}{(s+2)}+\frac{0.6}{s(s+2)}$$

(Initial Condition) + (Input)

Example 1 (cont)

- Use table to Invert $Y(s)$ term by term to find y(t)

$$Y(s)=\frac{1}{(s+2)}+\frac{0.6}{s(s+2)}$$

- From the Table:

$$L\left(\frac{1}{s+a}\right)=e^{-at} \quad \Rightarrow \quad L^{-1}\left(\frac{1}{s+2}\right)=e^{-2t}$$

$$L\left(\frac{a}{s(s+a)}\right)=\left(1-e^{-at}\right) \quad \Rightarrow \quad L^{-1}\left(\frac{(0.3)2}{s(s+2)}\right)=0.3\left(1-e^{-2t}\right)$$

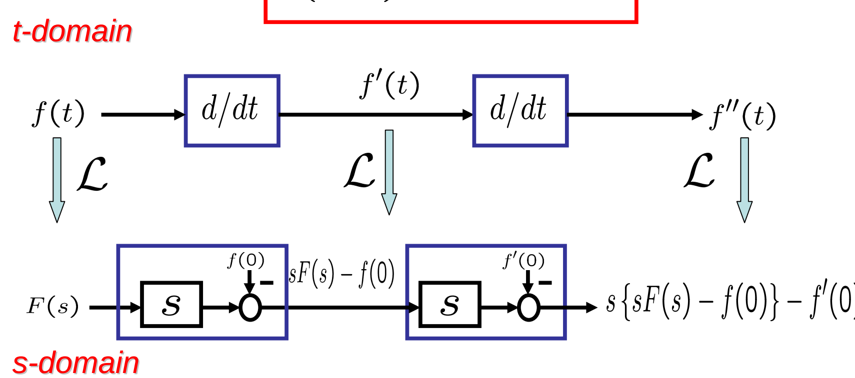
- So that

$$y(t)=e^{-2t}+0.3\left(1-e^{-2t}\right)$$

(Initial Condition) + (Input)

Properties of Laplace transform *Differentiation (review)*

$$\mathcal{L}\left\{f'(t)\right\}=sF(s)-f(0)$$



Example 2

$$\ddot{y}(t) - y(t) = t, \quad y(0) = 1, \dot{y}(0) = 1$$

$$s^2 Y(s) - sy(0) - \dot{y}(0) - Y(s) = \frac{1}{s^2},$$

$$s^2 Y(s) - s - 1 - Y(s) = \frac{1}{s^2} \Rightarrow (s^2 - 1)Y(s) = (s+1) + \frac{1}{s^2}$$

$$Y(s) = \frac{(s+1)}{(s^2-1)} + \frac{1}{s^2(s^2-1)}$$

$$Y(s) = \frac{A}{(s-1)} + \frac{B}{(s+1)} + \frac{C}{s^2} \quad \text{(Find A, B and C)}$$

How do we do that???

$$Y(s) = \frac{(3/2)}{(s-1)} + \frac{(-1/2)}{(s+1)} + \frac{(-1)}{s^2}$$

$$y(t) = (3/2)e^{t} + (-1/2)e^{-t} + (-1)t$$

Example 2 (cont'd)

Partial fraction expansion

unknowns

$$Y(s) = \frac{(s+1)}{(s^2-1)} + \frac{1}{s^2(s^2-1)} = \frac{A}{(s-1)} + \frac{B}{(s+1)} + \frac{C}{s^2}$$

Multiply both sides by (s-1) & let s ⇒ 1

$$(s-1)Y(s)\Big|_{s\to 1} = (s-1)\left[\frac{A}{(s-1)} + \frac{B}{(s+1)} + \frac{C}{s^2}\right] = A$$

$$\text{so} \quad A = (s-1)\left\{\frac{(s+1)}{(s^2-1)} + \frac{1}{s^2(s^2-1)}\right\}\Bigg|_{s\to 1} = \left\{1 + \frac{1}{s^2(s+1)}\right\} = 1 + \frac{1}{2} = \frac{3}{2}$$

Similarly,

$$B = (s+1)Y(s)\Big|_{s\to -1} = -\frac{1}{2}$$

$$C = (s^2)Y(s)\Big|_{s\to 0} = -1$$

Example 3

ODE with initial conditions (ICs)

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = 5u_s(t), \; y(0) = -1, \; y'(0) = 2$$

Laplace transform

$$\underbrace{s^2Y(s) - sy(0) - y'(0)}_{\mathcal{L}\{y''(t)\}} + 3\underbrace{\{sY(s) - y(0)\}}_{\mathcal{L}\{y'(t)\}} + 2Y(s) = \frac{5}{s}$$

$$\Rightarrow \quad Y(s) = \frac{-s^2 - s + 5}{s(s+1)(s+2)}$$

(This also isn't in the table…)

Example 3 (cont'd)

Inverse Laplace transform

$$\mathcal{L}^{-1}\left\{Y(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}\right\}$$

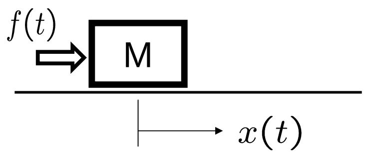
$$\Rightarrow \quad y(t) = \left(\underbrace{\frac{5}{2}}_{A} + \underbrace{(-5)}_{B} e^{-t} + \underbrace{\frac{3}{2}}_{C} e^{-2t}\right) u_s(t)$$

If we are interested in only the final value of y(t), apply Final Value Theorem:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{-s^2 - s + 5}{(s+1)(s+2)} = \frac{5}{2}$$

Example: Newton's law

$$M\frac{d^2x(t)}{dt^2} = f(t)$$



We want to know the trajectory of x(t). By Laplace transform,

$$M\left(s^2X(s) - sx(0) - x'(0)\right) = F(s)$$

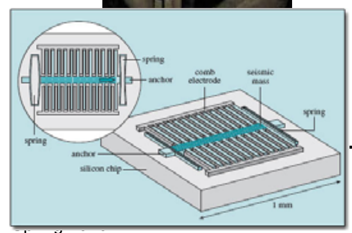
$$\Rightarrow X(s) = \frac{1}{Ms^2}F(s) + \frac{x(0)}{s} + \frac{x'(0)}{s^2}$$

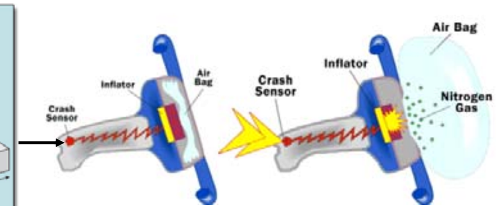
(Total response) = **(Forced response)** + **(Initial condition response)**

$$\Rightarrow x(t) = \mathcal{L}^{-1}\left[\frac{1}{Ms^2}F(s)\right] + x(0)u_s(t) + x'(0)tu_s(t)$$

EX. Air bag and accelerometer

- Tiny MEMS accelerometer
  - Microelectromechanical systems (MEMS)





(Pictures from various websites)

In this way, we can find a rather complicated solution to ODEs easily by using Laplace transform table!

Summary & Exercises

- Laplace transform (Important math tool!)
  - Definition
  - Laplace transform table
  - Properties of Laplace transform
  - Solution to ODEs via Laplace transform
- Exercises
  - Read Appendix A, B.
  - Solve Quiz problems…