What is Root Locus? (Review)

- Consider a feedback system that has one parameter (gain) $K > 0$ to be designed.

- **Root locus** graphically shows how poles of the closed-loop system varies as $K$ varies from 0 to infinity.

Characteristic equation & root locus

- **Characteristic equation**

$$1 + KL(s) = 0 \iff K = \frac{-1}{L(s)} \iff L(s) = \frac{-1}{K}$$

- **Root locus** is obtained by
  - for a fixed $K > 0$, finding roots of the characteristic equation, and
  - sweeping $K$ over real positive numbers.

- A point $s$ is on the root locus, if and only if $L(s)$ evaluated for that $s$ is a negative real number.
Angle and magnitude conditions

- Characteristic eq. can be split into two conditions.
  - **Angle condition**
    \[ \angle L(s) = 180^\circ \times (2k + 1), \ k = 0, \pm 1, \pm 2, \ldots \]
  - **Magnitude condition**
    \[ |L(s)| = \frac{1}{K} \]
    For any point \( s \), this condition holds for some positive \( K \).

A simple example

\[ L(s) = \frac{1}{s(s + 2)} \]

- Select a point \( s = -1+j \)
  \[ L(s) = \frac{1}{(-1+j)(1+j)} = -\frac{1}{2} \]
  \[ \angle L(s) = 180 \]
  \[ s \text{ is on root locus.} \]
  \[ K = \frac{1}{|L(s)|} = 2 \]

Root locus: Step 0

- **Root locus is symmetric w.r.t. the real axis.**
- Characteristic equation is an equation with real coefficients. Hence, if a complex number is a root, its complex conjugate is also a root.
- **The number of branches = order of \( L(s) \)**
  - If \( L(s) = n(s)/d(s) \), then Ch. eq. is \( d(s) + Kn(s) = 0 \), which has roots as many as the order of \( d(s) \).
- **Mark poles of \( L \) with “x” and zeros of \( L \) with “o”**.

\[ L(s) = \frac{s - z_1}{(s - p_1)(s - p_2)} \]

Root locus: Step 1-1

- **RL includes all points on real axis to the left of an odd number of real poles/zeros.**
  - Test point
    \[ \angle L(s) = \angle(s - z_1) - \angle(s - p_1) - \angle(s - p_2) \]
    \[ \text{Not satisfy angle condition!} \]
  - \[ \angle L(s) = 180 - \angle(s - z_1) - \angle(s - p_1) - \angle(s - p_2) \]
    \[ \text{Satisfy angle condition!} \]
Root locus: Step 1-1 (cont’d)

- RL includes all points on real axis to the left of an odd number of real poles/zeros.

\[ \ell L(s) = \ell (s - z_1) - \ell (s - p_1) - \ell (s - p_2) \]

\[ \text{Not satisfy angle condition!} \]

\[ \ell L(s) = \ell (s - z_1) - \ell (s - p_1) - \ell (s - p_2) \]

\[ \text{Satisfy angle condition!} \]

Root locus: Step 1-2

- RL originates from the poles of \( L \), and terminates at the zeros of \( L \), including infinity zeros.

\[ 1 + K \frac{n(s)}{d(s)} = 0 \Leftrightarrow d(s) + K n(s) = 0 \Leftrightarrow \frac{1}{K} + \frac{n(s)}{d(s)} = 0 \]

\[ K = 0 \quad \text{or} \quad K = \infty \]

\[ d(s) = 0 \quad \frac{n(s)}{d(s)} = 0 \]

\( s \): Poles of \( L(s) \) \quad \( s \): Zeros of \( L(s) \)

Root locus: Step 2-1

- Number of asymptotes = relative degree \( (r) \) of \( L \):
  \[ r := \deg (\text{den}) - \deg (\text{num}) \]

- Angles of asymptotes are
  \[ \frac{\pi}{r} \times (2k + 1), \ k = 0, 1, \ldots \]

\[ r = 1 \quad r = 2 \quad r = 3 \quad r = 4 \]

\[ \frac{\pi}{r} \] \quad \[ \frac{\pi}{2r} \] \quad \[ \frac{\pi}{3r} \] \quad \[ \frac{\pi}{4r} \]

Root locus: Step 2-1 (cont’d)

- For a very large \( s \),
  \[ L(s) = \frac{n_0 s^{n-r} + \cdots}{s^n + \cdots} \approx \frac{n_0}{s^r} \]

- Ch. eq is approximately
  \[ 1 + K L(s) = 0 \Rightarrow 1 + K \frac{n_0}{s^r} = 0 \Rightarrow s^r + K n_0 = 0 \]

\[ \Rightarrow s^r = -K n_0 < 0 \quad (\text{we assume } n_0 > 0) \]

\[ \Rightarrow \ell s^r = \pi \times (2k + 1), \ k = 0, 1, 2, \ldots \]

\[ \Rightarrow \ell s = \frac{\pi}{r} \times (2k + 1), \ k = 0, 1, 2, \ldots \]
Root locus: Step 2-2

- **Intersections of asymptotes**
  \[
  \sum_{r} \text{pole} - \sum_{r} \text{zero}
  \]

- Proof for this is omitted and not required in this course.
- Interested students should read page 363 in the book by Dorf & Bishop.

Root locus: Step 3

- **Breakaway points are among roots of**
  \[
  \frac{dL(s)}{ds} = 0
  \]

Suppose that \(s=b\) is a breakaway point.

\[
\begin{align*}
  d(b) + Kn(b) &= 0 \\
  d'(b) + Kn'(b) &= 0 \\
  \Rightarrow d'(b) - \frac{d(b)}{n(b)}n'(b) &= 0
\end{align*}
\]

\[
\frac{dL(s)}{ds} \bigg|_{s=b} = \frac{n'(b)d(b) - n(b)d'(b)}{d(b)^2} = -\frac{n(b)}{d^2(b)} \left( d'(b) - \frac{d(b)}{n(b)}n'(b) \right) = 0
\]

Root locus: Step 4

- **RL departs from a pole** \(p_j\) with **angle of departure**
  \[
  \theta_d = \sum_{i} (p_j - z_i) - \sum_{i,i\neq j} (p_j - p_i) + 180
  \]

- **RL arrives at a zero** \(z_j\) with **angle of arrival**
  \[
  \theta_a = \sum_{i} (z_j - p_i) - \sum_{i,i\neq j} (z_j - z_i) + 180
  \]
  
  *(No need to memorize these formula.)*

Root locus: Step 4 (cont’d)

- **Sketch of proof for angle of departure**

  For \(s\) to be on root locus, due to **angle condition**
  \[
  \phi_1 - \theta_1 - \theta_2 = 180
  \]
  \[
  |s - p_1| \rightarrow 0
  \]
  \[
  \angle(p_1 - z_1) - \theta_1 - \angle(p_1 - p_2) = 180
  \]
Root locus: Step 4 (cont’d)

- Sketch of proof for **angle of arrival**

For $s$ to be on root locus, due to **angle condition**

\[
\phi_1 + \phi_2 - \theta_1 - \theta_2 - \theta_3 = 180
\]

\[
|s - z_1| \to 0
\]

\[
\phi_1 + \angle(z_1 - z_2) - \sum_{i=1}^{3} \angle(z_1 - p_i) = 180
\]

Summary and exercises

- Sketch of proofs for root locus algorithm
- Next, we will move on to root locus applications to control examples.