What is Root Locus? (Review)

- Consider a feedback system that has one parameter (gain) $K > 0$ to be designed.

- Root locus graphically shows how poles of the closed-loop system varies as $K$ varies from 0 to infinity.

Root locus: Step 0 (Mark pole/zero)

- Root locus is symmetric w.r.t. the real axis.
- The number of branches = order of $L(s)$
- Mark poles of $L$ with “x” and zeros of $L$ with “o”.

$$L(s) = \frac{1}{s(s + 1)(s + 5)}$$
Root locus: Step 1 (Real axis)

- **RL includes all points on real axis to the left of an odd number of real poles/zeros.**
- **RL originates from the poles of L and terminates at the zeros of L, including infinity zeros.**

Root locus: Step 2 (Asymptotes)

- **Number of asymptotes = relative degree (r) of L:**
  \[ r := \text{deg (den)} - \text{deg (num)} \]
- **Angles of asymptotes are**
  \[ \frac{\pi}{r} \times (2k + 1), \quad k = 0, 1, \ldots \]

  \[ r = 1 \quad r = 2 \quad r = 3 \quad r = 4 \]

Root locus: Step 2 (Asymptotes) - Intersections of asymptotes

- **Intersections of asymptotes**
  \[ \sum \text{pole} - \sum \text{zero} \]
  \[ L(s) = \frac{1}{s(s + 1)(s + 5)} \Rightarrow \sum \text{pole} - \sum \text{zero} = \frac{r}{3} = -2 \]

Root locus: Step 3 (Breakaway)

- **Breakaway points are among roots of**
  \[ \frac{dL(s)}{ds} = 0 \]
  \[ L(s) = \frac{1}{s(s + 1)(s + 5)} \Rightarrow \frac{dL(s)}{ds} = -3s^2 + 12s + 5 = 0 \]

  \[ s = -2 \pm \frac{\sqrt{21}}{3} \]

  **For each candidate s, check the positivity of**
  \[ K = \frac{1}{L(s)} \]
  \[ s = -2 + \frac{\sqrt{21}}{3} \approx -0.47 \quad K \approx 1.13 \]
  \[ s = -2 - \frac{\sqrt{21}}{3} \approx -3.52 \quad K \approx -13.1 \]
Root locus: Step 3 (Breakaway)

\[ L(s) = \frac{1}{s(s + 1)(s + 5)} \]

What is this value? For what \( K \)?

- 5 - 2 - 1 - 0

What is this value? For what \( K \)?

Breakaway point

\( -0.47 \) \((K = 1.13)\)

Finding K for critical stability

- Characteristic equation

\[ 1 + \frac{K}{s(s + 1)(s + 5)} = 0 \iff s^3 + 6s^2 + 5s + K = 0 \]

- Routh array

<table>
<thead>
<tr>
<th>( s )</th>
<th>1</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s^2 )</td>
<td>6</td>
<td>( K )</td>
</tr>
<tr>
<td>( s^1 )</td>
<td>( 30 - K )</td>
<td></td>
</tr>
<tr>
<td>( s^0 )</td>
<td>( K \</td>
<td></td>
</tr>
</tbody>
</table>

Stability condition

\[ 0 < K < 30 \]

- When \( K = 30 \)

\[ 6s^2 + 30 = 0 \implies s = \pm \sqrt{5}j \]

Root locus

\[ L(s) = \frac{1}{s(s + 1)(s + 5)} \]

\( \sqrt{5}j \)

(K = 30)

Breakaway point

\( -0.47 \) \((K = 1.13)\)

Example with complex poles

\[ L(s) = \frac{s}{s^2 + s + 1} \]

After Steps 0, 1, 2, 3, we obtain

zero 0
pole \( -\frac{1}{2} \pm j\frac{\sqrt{3}}{2} \)

How to compute angle of departure?

Breakaway point

\[ s^2 + s + 1 - s(2s + 1) = 0 \]

\[ \implies s = \pm 1 \]
**Root locus: Step 4**  
**Angle of departure**
- **Angle condition:** For $s$ to be on RL,

\[
\frac{\theta_1}{\theta_2} = \frac{s-z_1}{s-p_1(s-p_2)} = \frac{\angle(s-z_1) - \angle(s-p_1) - \angle(s-p_2)}{}
\]

If $s$ is close to $p_1$:
- $\phi_1 \approx 120$, $\theta_2 \approx 90$
  \[\Rightarrow \theta_1 = -150\]

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**Summary and exercises**
- **Examples for root locus.**
  - Gain computation for marginal stability, by using Routh-Hurwitz criterion
  - Angle of departure (Angle of arrival can be obtained by a similar argument.)
- **Next, sketch of proofs for root locus algorithm**
- **Exercises**
  - Draw root locus for $K > 0$ (no need to consider $K < 0$) for open-loop transfer functions in
    - Problems 7.5 and 7.7.

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**Exercises 1**

\[
L(s) = \frac{s}{s^2 + s + 1} \quad L(s) = \frac{s + 1}{s^2}
\]

\[
L(s) = \frac{1}{(s-1)(s+2)(s+3)} \quad L(s) = \frac{s}{(s+1)(s^2 + 1)}
\]
Exercises 2

\[ L(s) = \frac{1}{s(s + 1)(s + 2)} \quad L(s) = \frac{1}{(s + 1)(s^2 + 2s + 2)} \]

\[ L(s) = \frac{1}{s(s + 2)(s^2 + 2s + 2)} \quad L(s) = \frac{1}{(s^2 + 4s + 5)(s^2 + 2s + 5)} \]

Exercises 3

\[ L(s) = \frac{1}{s(s + 3)(s^2 + 2s + 2)} \quad L(s) = \frac{1}{s(s + 1)(s + 2)(s^2 + 2s + 2)} \]

\[ L(s) = \frac{s + 1}{s^2 + 4s + 5} \quad L(s) = \frac{s + 3}{(s + 1)(s^2 + 4s + 5)} \]

Exercises 4

\[ L(s) = \frac{s + 4}{(s + 1)(s + 3)(s^2 + 4s + 5)} \quad L(s) = \frac{s + 2}{(s^2 + 2s + 5)(s^2 + 6s + 10)} \]

\[ L(s) = \frac{s^2 + 2s + 2}{s(s + 2)(s + 3)} \quad L(s) = \frac{(s + 2)(s + 3)}{s(s + 1)} \]