Lecture plan

- L16: Root locus, sketching algorithm
- L17: Root locus, examples
- L18: Root locus, proofs
- L19: Root locus, control examples
- L20: Root locus, influence of zero and pole
- L21: Root locus, lead lag controller design

What is Root Locus?

- W. R. Evans developed in 1948.
- **Pole location** of the feedback system characterizes stability and transient properties.
- Consider a feedback system that has one parameter (gain) K>0 to be designed.

\[
\begin{array}{c}
K \\
\downarrow \\
L(s) \\
\end{array}
\]

\(L(s): \text{open-loop TF}\)

- **Root locus** graphically shows how poles of CL system vary as K varies from 0 to infinity.
A simple example

Characteristic eq. \(1 + K \frac{1}{s(s + 2)} = 0\)

\[s^2 + 2s + K = 0 \quad \rightarrow \quad s = -1 \pm \sqrt{1 - K}\]

- \(K=0: s=0,-2\)
- \(K=1: s=-1,-1\)
- \(K>1: \text{complex numbers}\)

A more complicated example

Characteristic eq. \(1 + K \frac{s + 1}{s(s + 2)(s + 3)} = 0\)

\[s(s + 2)(s + 3) + K(s + 1) = 0 \quad \rightarrow \quad s = \text{???}\]

- It is hard to solve this analytically for each \(K\).
- Is there some way to sketch roughly root locus by hand? (In Matlab, use command “rlocus.m”.)

Root locus: Step 0

- Root locus is symmetric w.r.t. the real axis.
- The number of branches = order of \(L(s)\)
- Mark poles of \(L\) with “\(x\)” and zeros of \(L\) with “\(o\)”.

\[L(s) = \frac{s + 1}{s(s + 2)(s + 3)}\]

Root locus: Step 1

- RL includes all points on real axis to the left of an odd number of real poles/zeros.
- RL originates from the poles of \(L\) and terminates at the zeros of \(L\), including infinity zeros.

Indicate the direction with an arrowhead.
Root locus: Step 2 (Asymptotes)

- **Number of asymptotes** = relative degree ($r$) of $L$:
  \[ r := \frac{n}{\deg(\text{den})} - \frac{m}{\deg(\text{num})} \]

- **Angles of asymptotes** are
  \[ \frac{\pi}{r} \times (2k + 1), \ k = 0, 1, \ldots \]

- $r = 1$  
- $r = 2$  
- $r = 3$  
- $r = 4$

Root locus: Step 2 (Asymptotes)

- **Intersections of asymptotes**
  \[ L(s) = \frac{s + 1}{s(s + 2)(s + 3)} \]
  \[ \sum \frac{\text{pole} - \sum \text{zero}}{r} = \frac{(r + 2) - (3)}{2} = -2 \]

Asymptotes (Not root locus)

Root locus: Step 3

- **Breakaway points** are among roots of
  \[ \frac{dL(s)}{ds} = 0 \]

Points where two or more branches meet and break away.

\[ L(s) = \frac{s + 1}{s(s + 2)(s + 3)} \]
\[ \frac{dL(s)}{ds} = -2s^3 + 4s^2 + 5s + 3 = 0 \]

\[ s = -2.4656, -0.7672 \pm 0.7926i \]

For each candidate $s$, check the positivity of $K = -\frac{1}{L(s)}$

\[ K = 0.4186, 1.7907 \pm 4.2772i \]

Quotient rule

\[ \left( \frac{N}{D} \right)' = \frac{N'D - ND'}{D^2} \]

\[ \left( \frac{s + 1}{s(s + 2)(s + 3)} \right)' = \frac{s(s^2 + 5s + 6) - (s + 1)(3s^2 + 10s + 6)}{(s(s + 2)(s + 3))^2} = \frac{-2s^3 - 8s^2 - 10s - 6}{(s(s + 2)(s + 3))^2} \]
Root locus: Step 3

Breakaway point

-2.46

$K = 0.4186$

Matlab command “rlocus.m”

```matlab
num=[1 1];
den=[1 5 6 0];
sys=tf(num,den);
rlocus(sys)
```

A simple example: revisited

$L(s) = \frac{1}{s(s + 2)}$

- Asymptotes
  - Relative degree 2
  - Intersection $\frac{0 + (-2)}{2} = -1$
- Breakaway point
  
  $L'(s) = \frac{-(2s + 2)}{s} = 0 \Rightarrow s = -1$

Summary and exercises

- Root locus
  - What is root locus
  - How to roughly sketch root locus
- Sketching root locus relies heavily on experience. **PRACTICE!**
- To accurately draw root locus, use Matlab.
- Next, more examples
- Exercises
  - Read Chapter 7.
Exercises

\[ L(s) = \frac{1}{s} \quad L(s) = \frac{1}{s^2} \quad L(s) = \frac{1}{s^3} \]

\[ L(s) = \frac{1}{s(s + 4)} \quad L(s) = \frac{s + 1}{s(s + 2)} \quad L(s) = \frac{1}{s(s + 1)(s + 5)} \]