ME451: Control Systems

Lecture 16

Root locus

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Course roadmap

Modeling

- Laplace transform
- Transfer function
- Models for systems
  - electrical
  - mechanical
  - electromechanical
- Block diagrams
- Linearization

Analysis

- Time response
  - Transient
  - Steady state
- Frequency response
  - • Bode plot
- Stability
  - Routh-Hurwitz
  - • Nyquist

Design

- Design specs
- Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

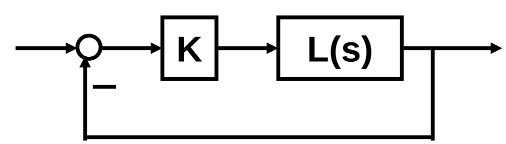
(Matlab simulations &) laboratories

Lecture plan

- L16: Root locus, sketching algorithm
- L17: Root locus, examples
- L18: Root locus, proofs
- L19: Root locus, control examples
- L20: Root locus, influence of zero and pole
- L21: Root locus, lead lag controller design

What is Root Locus?

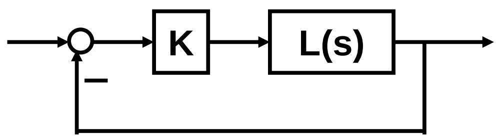
- W. R. Evans developed in 1948.
- *Pole location* of the feedback system characterizes stability and transient properties.
- Consider a feedback system that has one parameter (gain) K>0 to be designed.



L(s): open-loop TF

- *Root locus* graphically shows how poles of CL system varies as K varies from 0 to infinity.

A simple example



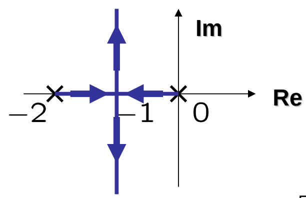
$$L(s) = \frac{1}{s(s+2)}$$

- Characteristic eq. $1 + K\frac{1}{s(s+2)} = 0$

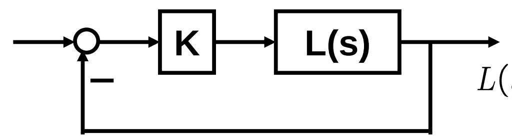
Closed-loop poles

$$\Rightarrow s^2 + 2s + K = 0 \Rightarrow s = -1 \pm \sqrt{1-K}$$

- K=0: s=0,-2
- K=1: s=-1,-1
- K>1: complex numbers



A more complicated example



$$L(s) = \frac{s+1}{s(s+2)(s+3)}$$

- Characteristic eq. $1 + K\frac{s+1}{s(s+2)(s+3)} = 0$

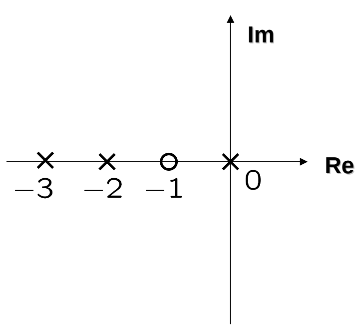
$$\Rightarrow s(s+2)(s+3) + K(s+1) = 0 \Rightarrow s = ???$$

- It is hard to solve this analytically for each K.
- Is there some way to sketch roughly root locus by hand? (In Matlab, use command "*rlocus.m*".)

Root locus: Step 0

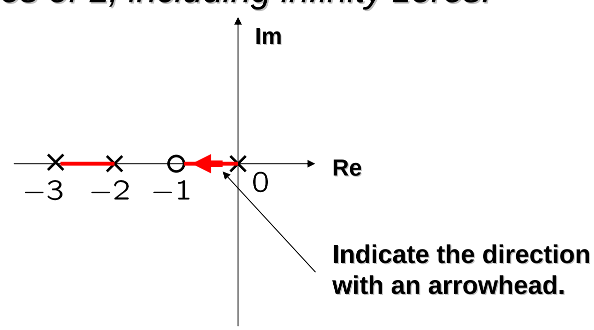
- *Root locus is symmetric w.r.t. the real axis.*
- *The number of branches = order of L(s)*
- *Mark poles of L with "x" and zeros of L with "o".*

$$L(s) = \frac{s+1}{s(s+2)(s+3)}$$



Root locus: Step 1

- *RL includes all points on real axis to the left of an odd number of real poles/zeros.*
- *RL originates from the poles of L and terminates at the zeros of L, including infinity zeros.*



Root locus: Step 2 (Asymptotes)

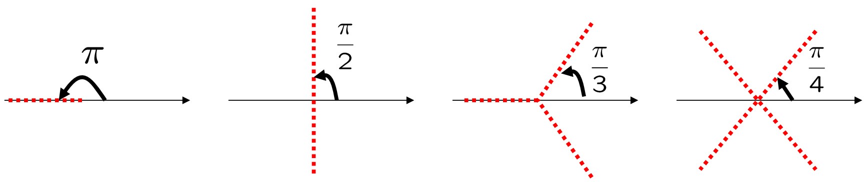
- *Number of asymptotes = relative degree (r) of L:*

$$r := \underbrace{n}_{\deg(\text{den})} - \underbrace{m}_{\deg(\text{num})}$$

- *Angles of asymptotes are*

$$\frac{\pi}{r} \times (2k+1),\ k = 0, 1, \ldots$$

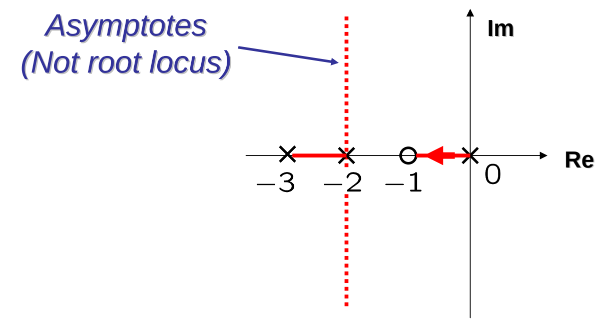
$r=1$ $\quad$ $r=2$ $\quad$ $r=3$ $\quad$ $r=4$



Root locus: Step 2 (Asymptotes)

- *Intersections of asymptotes* $\dfrac{\sum \text{pole} - \sum \text{zero}}{r}$

$$L(s) = \frac{s+1}{s(s+2)(s+3)} \Rightarrow \frac{\sum \text{pole} - \sum \text{zero}}{r} = \frac{(0+(-2)+(-3))-(-1)}{2} = -2$$



Root locus: Step 3

- *Breakaway points are among roots of* $\boxed{\dfrac{dL(s)}{ds} = 0}$

Points where two or more branches meet and break away.

$$L(s) = \frac{s+1}{s(s+2)(s+3)} \Rightarrow \frac{dL(s)}{ds} = -2\frac{s^3+4s^2+5s+3}{(*)} = 0$$

$$\Rightarrow s = -2.4656,\ -0.7672 \pm 0.7926i$$

For each candidate s, check the positivity of $K = -\dfrac{1}{L(s)}$

$$\Rightarrow K = 0.4186,\ 1.7907 \mp 4.2772i$$

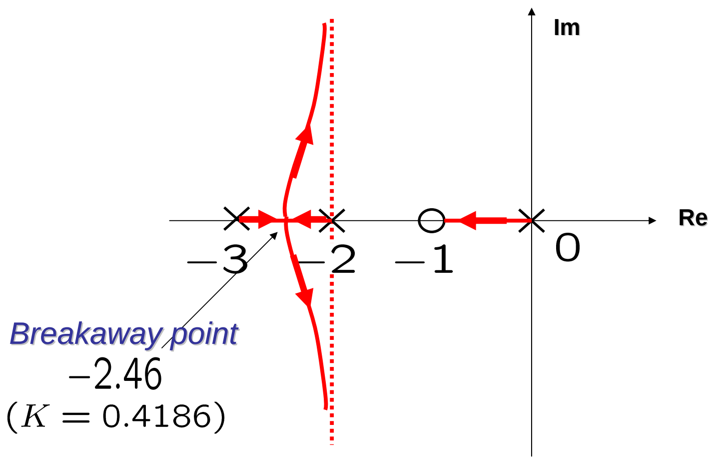
Quotient rule

$$\left(\frac{N}{D}\right)' = \frac{N'D - ND'}{D^2}$$

$$\left(\frac{s+1}{s(s+2)(s+3)}\right)' = \frac{s(s^2+5s+6) - (s+1)(3s^2+10s+6)}{(s(s+2)(s+3))^2}$$

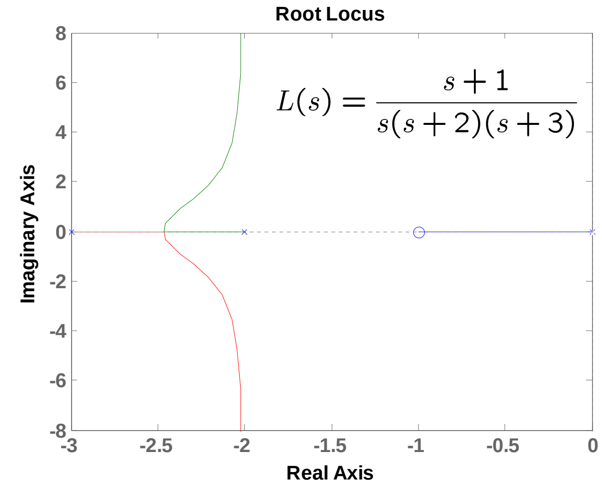
$$= \frac{-2s^3 - 8s^2 - 10s - 6}{(s(s+2)(s+3))^2}$$

Root locus: Step 3

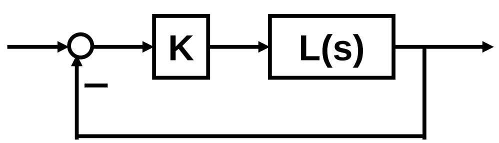


Matlab command “rlocus.m”

```
num=[1 1];
den=[1 5 6 0];
sys=tf(num,den);
rlocus(sys)
```

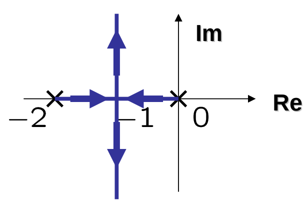


A simple example: revisited



$$L(s) = \frac{1}{s(s+2)}$$

- Asymptotes
  - Relative degree 2
  - Intersection $\frac{0+(-2)}{2} = -1$



- Breakaway point

$$L'(s) = \frac{-(2s+2)}{*} = 0 \Rightarrow s = -1$$

Summary and exercises

- Root locus
  - What is root locus
  - How to roughly sketch root locus
- Sketching root locus relies heavily on experience. **PRACTICE!**
- To accurately draw root locus, use Matlab.
- Next, more examples
- Exercises
  - Read Chapter 7.

Exercises

$$L(s) = \frac{1}{s} \qquad L(s) = \frac{1}{s^2} \qquad L(s) = \frac{1}{s^3}$$

$$L(s) = \frac{1}{s(s+4)} \qquad L(s) = \frac{s+1}{s(s+2)} \qquad L(s) = \frac{1}{s(s+1)(s+5)}$$