Lecture 15
Time response of 2nd-order systems

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Performance measures (review)

- Transient response
  - Peak value
  - Peak time
  - Percent overshoot
  - Delay time
  - Rise time
  - Settling time
- Steady state response
  - Steady state error

Next, we will connect these measures with s-domain.

Second-order systems

- A standard form of the second-order system

\[ G(s) = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2} \]

\[ \zeta : \text{damping ratio} \]
\[ \omega_n : \text{undamped natural frequency} \]

- DC motor position control example

\[ R(s) \rightarrow K \rightarrow E_a(s) \rightarrow \frac{0.5}{s(s+2)} \rightarrow \Theta_m(s) \rightarrow \text{Motor} \]

Closed-loop TF

\[ \frac{\Theta_m(s)}{R(s)} = \frac{0.5K}{s^2 + 2s + 0.5K} \]
Step response for 2nd-order system

- Input a unit step function to a 2nd-order system. What is the output?

\[ y(t) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \]

DC gain

\[ G(0) = 1 \rightarrow \lim_{t \to \infty} y(t) = G(0) = 1 \text{ if } G \text{ is stable} \]

Step response for 2nd-order system for various damping ratio

- Undamped \( \zeta = 0 \)
- Underdamped \( 0 < \zeta < 1 \)
- Critically damped \( \zeta = 1 \)
- Overdamped \( \zeta > 1 \)

Step response for 2nd-order system

Underdamped case

- Math expression of \( y(t) \) for underdamped case

\[ Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \frac{1}{s} \]

\[ \mathcal{L}^{-1} \left[ y(t) \right] = 1 - \frac{e^{-\zeta \omega_n t}}{\sqrt{1 - \zeta^2}} \sin \left( \omega_d t + \cos^{-1} \zeta \right) \]

Damped natural frequency

\[ \omega_d := \omega_n \sqrt{1 - \zeta^2} \]

Peak value/time: Underdamped case

\[ y(t) = 1 - \frac{e^{-\zeta \omega_n t}}{\sqrt{1 - \zeta^2}} \sin \left( \omega_d t + \cos^{-1} \zeta \right) \]

\[ y_{\text{max}} = 1 + e^{\frac{-\zeta \pi}{\sqrt{1 - \zeta^2}}} \]

\[ y(0) = 0 \]

\[ y'(0) = 0 \]
Properties of 2nd-order system

\[ G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2} \quad 0 < \zeta < 1 \]

<table>
<thead>
<tr>
<th>Property</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>Peak time</td>
<td>( \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} )</td>
</tr>
<tr>
<td>Peak value</td>
<td>( 1 + e^{-\zeta\pi/\sqrt{1-\zeta^2}} )</td>
</tr>
<tr>
<td>Percent overshoot</td>
<td>( 100e^{-\zeta\pi/\sqrt{1-\zeta^2}} )</td>
</tr>
<tr>
<td>Settling time</td>
<td>( \approx \frac{3}{\zeta\omega_n} ) or ( \frac{4}{\zeta\omega_n} ) (5%) (2%)</td>
</tr>
</tbody>
</table>

Some remarks

- Percent overshoot depends on \( \zeta \), but NOT \( \omega_n \).
- From 2nd-order transfer function, analytic expressions of delay & rise time are hard to obtain.
- Time constant is \( 1/(\zeta\omega_n) \), indicating convergence speed.
- For \( \zeta > 1 \), we cannot define peak time, peak value, percent overshoot.

P.O. vs. damping ratio

\[ 100e^{-\zeta\pi/\sqrt{1-\zeta^2}} \]

Pole locations of G

- Poles (0<\( \zeta <1 \))
  \[ s = -\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \]
- Damping ratio
  \[ \zeta = \cos \theta \]

Next, we clarify the influence of pole location on step response.
Influence of real part of poles

- Settling time $t_{ss}$ decreases.

\[
y(t) = 1 - e^{-\zeta \omega_d t} \sin(\omega_d t + \cos^{-1} \zeta)\]

Influence of imag. part of poles

- Oscillation frequency $\omega_d$ increases.

\[
y(t) = 1 - e^{-\zeta \omega_d t} \sqrt{1 - \zeta^2} \sin(\omega_d t + \cos^{-1} \zeta)\]

Influence of angle of poles

- Over/under-shoot decreases.

\[
y(t) = 1 - e^{-\zeta \omega_d t} \sin(\omega_d t + \cos^{-1} \zeta)\]

\[\zeta = \cos \theta\]

An example

- Require 5% settling time $t_s < t_{sm}$ (given):

\[
t_s \approx \frac{3}{\zeta \omega_n} < t_{sm} \quad \text{for} \quad \zeta \omega_n > \frac{3}{t_{sm}}
\]

\[
\omega_n \sqrt{1 - \zeta^2} = \omega_d
\]

\[
\frac{3}{t_{sm}}
\]
An example (cont’d)

- Require PO < POm (given):
  \[ PO = 100e^{-\zeta\pi/\sqrt{1-\zeta^2}} < PO_m \]
  \[ = 100e^{-\pi/\tan\theta} \]

\[ \begin{array}{c|c}
  PO_m & \theta_m \\
  \hline
  4.3\% & \pi/4 \\
  16.3\% & \pi/3 \\
\end{array} \]

An example (cont’d)

- Combination of two requirements
  \[ \zeta\omega_n > \frac{3}{t_{sm}} \quad \& \quad \theta < \theta_m \]

Summary

- Transient response of 2nd-order system is characterized by
  - Damping ratio \( \zeta \) & undamped natural frequency \( \omega_n \)
  - Pole locations
- Delay time and rise time are not so easy to characterize, and thus not covered in this course.
- For transient responses of high order systems, we need computer simulations.
- Next, Root locus

Exercises

(Use a calculator if necessary.)

- Read Chapters 4.2 and 4.3.
- Solve Problem 4.11.

1. For the system below with \( \zeta = 0.6 \), \( \omega_n = 5 \) (rad/sec), obtain
   - Percent overshoot ?
   - 5% settling time ?
Exercises

2. For the system below, design $K_1$ and $K_2$ s.t.
   - Percent overshoot is at most 20%?
   - Peak time is at most 1 sec.?
   - With designed $K_1$ and $K_2$, what is 5% settling time?

![Block Diagram](image-url)