ME451: Control Systems

Lecture 10
Routh-Hurwitz stability criterion

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Course roadmap

Modeling
- Laplace transform
- Transfer function
- Models for systems
  - electrical
  - mechanical
  - electromechanical
- Linearization

Analysis
- Time response
  - Transient
  - Steady state
- Frequency response
  - Bode plot
- Stability
  - Routh-Hurwitz
  - Nyquist

Design
- Design specs
- Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

(Matlab simulations & laboratories)

Stability summary (review)

Let $s_i$ be poles of rational $G$. Then, $G$ is …

- (BIBO, asymptotically) stable if $\text{Re}(s_i)<0$ for all $i$.
- marginally stable if
  - $\text{Re}(s_i)<=0$ for all $i$, and
  - simple root for $\text{Re}(s_i)=0$
- unstable if it is neither stable nor marginally stable.

Routh-Hurwitz criterion

- This is for LTI systems with a polynomial denominator (without sin, cos, exponential etc.)
- It determines if all the roots of a polynomial
  - lie in the open LHP (left half-plane),
  - or equivalently, have negative real parts.
- It also determines the number of roots of a polynomial in the open RHP (right half-plane).
- It does NOT explicitly compute the roots.
- No proof is provided in any control textbook.
Polynomial and an assumption

- Consider a polynomial
  \[ Q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 \]

- Assume \( a_0 \neq 0 \)
  - If this assumption does not hold, \( Q \) can be factored as
    \[ Q(s) = s^m \left( \tilde{a}_{n-m} s^{n-m} + \cdots + \tilde{a}_1 s + \tilde{a}_0 \right) \]
    where \( \tilde{a}_0 \neq 0 \)
  - The following method applies to the polynomial \( \tilde{Q}(s) \)

Routh array

- From the given polynomial

### Routh array

| \( s^n \) | \( a_n \) | \( a_{n-2} \) | \( a_{n-4} \) | \( a_{n-6} \) | \( \cdots \) |
| \( s^{n-1} \) | \( a_{n-1} \) | \( a_{n-3} \) | \( a_{n-5} \) | \( a_{n-7} \) | \( \cdots \) |
| \( s^{n-2} \) | \( b_1 \) | \( b_2 \) | \( b_3 \) | \( b_4 \) | \( \cdots \) |
| \( s^{n-3} \) | \( c_1 \) | \( c_2 \) | \( c_3 \) | \( c_4 \) | \( \cdots \) |
| \( s^2 \) | \( k_1 \) | \( k_2 \) |
| \( s^1 \) | \( l_1 \) |
| \( s^0 \) | \( m_1 \) |

**Routh array (How to compute the third row)**

\[ b_1 = \frac{a_{n-2} a_{n-1} - a_n a_{n-3}}{a_{n-1}} \]
\[ b_2 = \frac{a_{n-4} a_{n-1} - a_n a_{n-5}}{a_{n-1}} \]

**Routh array (How to compute the fourth row)**

\[ c_1 = \frac{a_{n-3} b_1 - a_{n-1} b_2}{b_1} \]
\[ c_2 = \frac{a_{n-5} b_1 - a_{n-1} b_3}{b_1} \]

\[ \vdots \]
Routh-Hurwitz criterion

The number of roots in the open right half-plane is equal to the number of sign changes in the first column of the Routh array.

Example 1

\[ Q(s) = s^3 + s^2 + 2s + 8 = (s + 2)(s^2 - s + 4) \]

Routh array

\[ \begin{array}{cccc}
  s^3 & s^2 & s^1 & s^0 \\
  1 & -6 & 8 & \\
  s^2 & k_1 & k_2 & \\
  s^1 & l_1 & & \\
  s^0 & m_1 & & \\
\end{array} \]

Two sign changes in the first column \(1 \rightarrow -6 \rightarrow 8\) \(\Rightarrow\) Two roots in RHP \(\frac{1}{2} \pm \frac{j\sqrt{15}}{2}\)

Example 2

\[ Q(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10 \]

Routh array

\[ \begin{array}{cccc}
  s^5 & s^4 & s^3 & s^2 \\
  1 & 2 & 2 & 4 \\
  s^4 & k_1 & k_2 & \\
  s^3 & l_1 & & \\
  s^2 & m_1 & & \\
\end{array} \]

Two sign changes in the first column \(\Rightarrow\) Two roots in RHP

Example 3

\[ Q(s) = s^4 + s^3 + 3s^2 + 2s + 2 \]

Routh array

\[ \begin{array}{cccc}
  s^4 & s^3 & s^2 & s^1 \\
  1 & 1 & 3 & 2 \\
  s^3 & & & \\
  s^2 & & & \\
  s^1 & & & \\
  s^0 & & & \\
\end{array} \]

If zero row appears in the Routh array, \(Q\) has roots either on the imaginary axis or in RHP.

Take derivative of an auxiliary polynomial \(s^2 + 2\) \(\Rightarrow\) But some roots are on imag. axis.
Example 4

\[ Q(s) = s^3 + 3Ks^2 + (K + 2)s + 4 \]

Find the range of \( K \) s.t. \( Q(s) \) has all roots in the left half plane. (Here, \( K \) is a design parameter.)

**Routh array**

<table>
<thead>
<tr>
<th>( s^3 )</th>
<th>( s^2 )</th>
<th>( s^1 )</th>
<th>( s^0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 3K &gt; 0 )</td>
<td>( 3K(K + 2) - 4 &gt; 0 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( K &gt; -1 + \frac{\sqrt{21}}{3} )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Simple & important criteria for stability

- **1st order polynomial** \( Q(s) = a_1s + a_0 \)
  
  All roots are in LHP \( \iff a_1 \) and \( a_0 \) have the same sign

- **2nd order polynomial** \( Q(s) = a_2s^2 + a_1s + a_0 \)
  
  All roots are in LHP \( \iff a_2, a_1 \) and \( a_0 \) have the same sign

- **Higher order polynomial** \( Q(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0 \)
  
  All roots are in LHP \( \Rightarrow \) All \( a_k \) have the same sign

Examples

<table>
<thead>
<tr>
<th>( Q(s) )</th>
<th>All roots in open LHP?</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 3s + 5 )</td>
<td>Yes / No</td>
</tr>
<tr>
<td>( -2s^2 - 5s - 100 )</td>
<td>Yes / No</td>
</tr>
<tr>
<td>( 523s^2 - 57s + 189 )</td>
<td>Yes / No</td>
</tr>
<tr>
<td>( (s^2 + s - 1)(s^2 + s + 1) )</td>
<td>Yes / No</td>
</tr>
<tr>
<td>( s^3 + 5s^2 + 10s - 3 )</td>
<td>Yes / No</td>
</tr>
</tbody>
</table>

Summary and Exercises

- Routh-Hurwitz stability criterion
  - Routh array
  - Routh-Hurwitz criterion is applicable to only polynomials (so, it is not possible to deal with exponential, sin, cos etc.).

- Next,
  - Routh-Hurwitz criterion in control examples

- Exercises
  - Read Section 6.
  - Do Examples and Problems 6-2.