ME451: Control Systems

Lecture 9
Stability

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Course roadmap

**Modeling**
- Laplace transform
- Transfer function
- Models for systems
  - electrical
  - mechanical
  - electromechanical
- Block diagrams
- Linearization

**Analysis**
- Time response
  - Transient
  - Steady state
- Frequency response
  - Bode plot
- Stability
  - Routh-Hurwitz
  - (Nyquist)

**Design**
- Design specs
- Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

(Matlab simulations & laboratories)
Simple mechanical examples

- We want mass to stay at $x=0$, but wind gave some initial speed ($F(t)=0$). What will happen?

![Diagrams showing mechanical systems with transfer functions $X(s)/F(s)$ for different configurations.]

- How to characterize different behaviors with TF?

Stability

- Utmost important specification in control design!
- Unstable systems have to be stabilized by feedback.
- Unstable closed-loop systems are useless.
  - What happens if a system is unstable?
    - may hit mechanical/electrical “stops” (saturation)
    - may break down or burn out
What happens if a system is unstable?

Tacoma Narrows Bridge (July 1-Nov.7, 1940)

Wind-induced vibration

Collapsed!

2008...

Mathematical definitions of stability

- **BIBO (Bounded-Input-Bounded-Output)** stability:
  Any bounded input generates a bounded output.

- **Asymptotic stability**:
  Any ICs generates $y(t)$ converging to zero.
Some terminologies

\[ G(s) = \frac{n(s)}{d(s)} \]

Ex. \[ G(s) = \frac{(s - 1)(s + 1)}{(s + 2)(s^2 + 1)} \]

- **Zero**: roots of \( n(s) \)  
  (Zeros of \( G \)) = \pm 1

- **Pole**: roots of \( d(s) \)  
  (Poles of \( G \)) = -2, \pm j

- **Characteristic polynomial**: \( d(s) \)

- **Characteristic equation**: \( d(s) = 0 \)

Stability condition in s-domain

(Proof omitted, and not required)

For a system represented by a transfer function \( G(s) \),

**system is BIBO stable**

*All the poles of \( G(s) \) are in the open left half of the complex plane.*

**system is asymptotically stable**
“Idea” of stability condition

Example \[ y'(t) + \alpha y(t) = u(t), \quad y(0) = y_0 \]

\[ sY(s) - y(0) + \alpha Y(s) = U(s) \]

\[ Y(s) = \frac{1}{s + \alpha} (U(s) + y(0)) \]

Asym. Stability: \((U(s)=0)\)

\[ y(t) = L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{1}{s + \alpha}y(0)\right\} = e^{-\alpha t}y(0) \rightarrow 0 \Leftrightarrow Re(\alpha) > 0 \]

BIBO Stability: \((y(0)=0)\)

\[ y(t) = L^{-1}\{Y(s)\} = L^{-1}\{G(s)U(s)\} = \int_0^t g(\tau)u(t-\tau)d\tau = \int_0^t e^{-\alpha \tau}u(t-\tau)d\tau \]

\[ |y(t)| \leq \int_0^t |e^{-\alpha \tau}||u(t-\tau)||d\tau \leq \int_0^t |e^{-\alpha \tau}|d\tau \cdot u_{max} \quad \text{Bounded if } Re(\alpha)>0 \]

Remarks on stability

- For a general system (nonlinear etc.), BIBO stability condition and asymptotic stability condition are different.
- For linear time-invariant (LTI) systems (to which we can use Laplace transform and we can obtain a transfer function), the conditions happen to be the same.
- In this course, we are interested in only LTI systems, we use simply “stable” to mean both BIBO and asymptotic stability.
Remarks on stability (cont’d)

- **Marginally stable** if
  - \( G(s) \) has no pole in the open RHP (Right Half Plane), &
  - \( G(s) \) has at least one simple pole on \( j\omega \)-axis, &
  - \( G(s) \) has no multiple poles on \( j\omega \)-axis.

\[
G(s) = \frac{1}{s(s^2 + 4)(s + 1)} \quad \text{and} \quad G(s) = \frac{1}{s(s^2 + 4)^2(s + 1)}
\]

- **Unstable** if a system is neither stable nor marginally stable.

Examples

- **Repeated poles**

\[
\mathcal{L}^{-1}\left[\frac{2\omega}{(s^2 + \omega^2)^2}\right] = t \sin \omega t \quad \mathcal{L}^{-1}\left[\frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}\right] = t \cos \omega t
\]

\[
\cdots = t^2 \sin \omega t \quad \cdots = t^2 \cos \omega t
\]

- **Does marginal stability imply BIBO stability?**

- **TF:**
  \[
  G(s) = \frac{2s}{(s^2 + 1)}
  \]

- **Pick**
  \[
  u(t) = \sin t \quad \mathcal{L}^{-1}(U(s)) = \frac{1}{(s^2 + 1)}
  \]

- **Output**

\[
\mathcal{L}^{-1}\left[Y(s) = G(s)U(s) = \frac{2s}{(s^2 + 1)^2}\right] = t \sin t
\]
Stability summary

Let \( s_i \) be poles of \( G \).
Then, \( G \) is …

- **(BIBO, asymptotically) stable** if \( \text{Re}(s_i) < 0 \) for all \( i \).
- **marginally stable** if
  - \( \text{Re}(s_i) \leq 0 \) for all \( i \), and
  - simple root for \( \text{Re}(s_i) = 0 \)
- **unstable** if
  it is neither stable nor marginally stable.

Mechanical examples: revisited

\[
\frac{X(s)}{F(s)} = \frac{1}{s^2}
\]

\[
\frac{X(s)}{F(s)} = \frac{1}{s^2 + B}
\]

\[
\frac{X(s)}{F(s)} = \frac{1}{s^2 + B + K}
\]
Examples

\[
G(s) \quad \text{Stable/marginally stable/unstable}
\]

\[
\begin{align*}
\frac{5(s + 2)}{(s + 1)(s^2 + s + 1)} & \quad ? \\
\frac{5(-s + 2)}{(s + 1)(s^2 + s + 1)} & \quad ? \\
\frac{5}{(s - 2)(s^2 + 3)} & \quad ? \\
\frac{s^2 + 3}{(s + 1)(s^2 - s + 1)} & \quad ? \\
\frac{1}{(s + 1)(s^2 + 1)^2} & \quad ? \\
\frac{1}{(s^2 - 1)(s + 1)} & \quad ???
\end{align*}
\]

Summary and Exercises

- Stability for LTI systems
  - (BIBO and asymptotically) stable, marginally stable, unstable
  - Stability for \(G(s)\) is determined by poles of \(G\).

- Next
  - Routh-Hurwitz stability criterion to determine stability without explicitly computing the poles of a system.

- Exercises
  - Read Sections 5-1, 5-2, 5-5.
  - Solve examples in the previous slide.