Lecture 8
Modeling of DC motors

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Course roadmap

**Modeling**
- Laplace transform
- Transfer function
- Models for systems
  - electrical
  - mechanical
  - electromechanical
- Block diagrams
- Linearization

**Analysis**
- Time response
  - Transient
  - Steady state
- Frequency response
  - Bode plot
- Stability
  - Routh-Hurwitz
  - Nyquist

**Design**
- Design specs
- Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

(Matlab simulations & laboratories)
What is DC motor?

An actuator, converting electrical energy into rotational mechanical energy

(You will see DC motor during Lab 1 and 4.)

Why DC motor?

- Advantages:
  - high torque
  - speed controllability
  - portability, etc.
- Widely used in control applications: robot, tape drives, printers, machine tool industries, radar tracking system, etc.
- Used for moving loads when
  - Rapid (microseconds) response is not required
  - Relatively low power is required
How does DC motor work?

Model of DC motor

“a”: armature
  \( e_a \): applied voltage
  \( i_a \): armature current

“b”: back EMF

 mechanical
  \( \theta \): angular position
  \( \omega \): angular velocity
  \( J \): rotor inertia
  \( B \): viscous friction
Modeling of DC motor: time domain

- **Armature circuit**
  \[ e_a(t) = R_a i_a(t) + L_a \frac{d i_a(t)}{dt} + e_b(t) \]

- **Connection between mechanical/electrical parts**
  - **Motor torque**
    \[ \tau(t) = K_T i_a(t) \]
  - **Back EMF**
    \[ e_b(t) = K_b \omega(t) \]

- **Mechanical load**
  \[ J \ddot{\theta}(t) = \tau(t) - B \dot{\theta}(t) - \tau_l(t) \]

- **Angular position**
  \[ \omega(t) = \dot{\theta}(t) \]

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Modeling of DC motor: s-domain

- **Armature circuit**
  \[ I_a(s) = \frac{1}{R_a + L_a s} \left( E_a(s) - E_b(s) \right) \]

- **Connection between mechanical/electrical parts**
  - **Motor torque**
    \[ T(s) = K_T I_a(s) \]
  - **Back EMF**
    \[ E_b(s) = K_b \Omega(s) \]

- **Mechanical load**
  \[ \Omega(s) = \frac{1}{J s + B} \left( T(s) - T_L(s) \right) \]

- **Angular position**
  \[ \Theta(s) = \frac{1}{s} \Omega(s) \]
### DC motor: Block diagram

![DC Motor Block Diagram](image)

### Useful formula for feedback

**Negative feedback system**

\[ Y(s) = G(s)(R(s) - H(s)Y(s)) \]

\[ Y(s) = (1 + G(s)H(s))Y(s) = G(s)R(s) \]

\[ \frac{Y(s)}{R(s)} = \frac{F_g}{1 - L_y} = \frac{G(s)}{1 + G(s)H(s)} \]

**Memorize this!**

\[
\begin{pmatrix}
G(s) \\
G(s)H(s)(-1)
\end{pmatrix}
\]

: forward gain

: loop gain
Ex: Derivation of transfer functions

Compute transfer functions from $R(s)$ to $Y(s)$.

DC motor: Transfer functions (TF)

\[
\frac{\Omega(s)}{E_a(s)} = \frac{K_T}{1 + \frac{K_T K_R}{(L_a s + R_a)(s + B)}} = \frac{K_T}{(L_a s + R_a)(s + B) + K_R K_T} =: G_1(s)
\]

\[
\frac{\Omega(s)}{I_L(s)} = -\frac{1}{s} \frac{L_a s + R_a}{1 + \frac{K_R K_T}{(L_a s + R_a)(s + B)}} = -\frac{L_a s + R_a}{(L_a s + R_a)(s + B) + K_R K_T} =: G_2(s)
\]

2nd order system

\[
\Omega(s) = G_1(s) E_a(s) + G_2(s) I_L(s)
\]

\[
\Theta(s) = \frac{1}{s} \Omega(s) = \frac{1}{s} (G_1(s) E_a(s) + G_2(s) I_L(s))
\]
DC motor: Transfer functions (cont’d)

Note: In many cases $L_a \ll R_a$. Then, an approximated TF is obtained by setting $L_a = 0$.

\[
\frac{\Omega(s)}{E_a(s)} = \frac{K_T}{(L_a s + R_a)(J s + B) + K_b K_T} \approx \frac{K_T}{R_a(J s + B) + K_b K_T}
\]

\[
= \frac{K}{T \, s + 1} \quad \left( K := \frac{K_T}{R_a B + K_b K_T}, \quad T = \frac{R_a I}{R_a B + K_b K_T} \right)
\]

2\textsuperscript{nd} order system $\quad \Longrightarrow \quad$ 1\textsuperscript{st} order system

\[
\frac{\Theta(s)}{E_a(s)} = \frac{K}{s(T \, s + 1)}
\]

Summary and Exercises

- Modeling of DC motor
  - What is DC motor and how does it work?
  - Derivation of a transfer function
  - Block diagram with feedback

- Next
  - Stability of linear control systems, one of the most important topics in feedback control

- Exercises
  - Read Section 2.7, 2.8.
  - Go over the derivation for DC motor transfer functions by yourself. Obtain $T(s)/E_a(s)$. 
Main message until this point

Many systems can be represented as transfer functions!

Using the transfer functions, .... (to be continued)